Statistics 581, Problem Set 2

Wellner; 10/5/2005

Reading: Chapter 1, especially pages 13 - 17; start reading chapter 2; Ferguson pages 1-25.
Due: Wednesday, October 12, 2005.

1. Suppose that Y is a random variable with $E(Y^2) < \infty$.
(a) Show that

$$Var(Y) = E\{Var(Y|X)\} + Var\{E(Y|X)\}\,;$$

i.e.

$$E(Y - EY)^2 = E\{(Y - E(Y|X))^2\} + E\{[E(Y|X) - E(Y)]^2\}\,.$$

(b) Interpret (a) geometrically.
(c) Suppose that $Y \sim \chi^2_n(\delta)$. Compute $E(Y)$ and $Var(Y)$.
Hint: Use $E(Y) = E\{E(Y|X)\}$ and (a).

2. Ferguson, ACILST, #1, page 11.
Let $X_1, X_2, \ldots$ be i.i.d. random variables with densities $f(x) = \alpha x^{-(\alpha+1)} 1_{(1,\infty)}(x)$.
(a) For what values of $\alpha > 0$ and $r > 0$ is it true that $n^{-1}X_n \rightarrow_r 0$?
(b) For what values of $\alpha > 0$ is it true that $n^{-1}X_n \rightarrow_{a.s.} 0$?
(c) If $X_1, X_2, \ldots$ are independent with X_n having density $f_n(x) = \alpha_n x^{-(\alpha_n+1)} 1_{(1,\infty)}(x)$ for $n = 1, 2, \ldots$, Find the limit of $n^{-2}EX_n^2$ when $\alpha_n = 2 + n^{-\gamma}$ for $\gamma \in \mathbb{R}$.

3. Suppose that: (i) $X \sim N_n(\mu, \Sigma)$ where Σ is of rank $k < n$;
(ii) Σ is a projection matrix (i.e. $\Sigma^2 = \Sigma$);
(iii) $\Sigma\mu = \mu$.
Show that $X'X \sim \chi^2_k(\delta)$ with $\delta = \mu'\mu$.

4. Ferguson, ACILST, #1, page 6:
(a) If $X_n \sim$ Beta$(1/n, 1/n)$ and $X \sim$ Bernoulli$(1/2)$, show that $X_n \rightarrow_d X$.
(b) Suppose that $X_n \sim$ Beta$(\alpha/n, \beta/n)$ with $\alpha, \beta > 0$. Does $X_n \rightarrow_d X$ for some X?

5. (a) Ferguson, ACILST, #4, page 6: Give an example of random variables X_n such that $E|X_n| \to 0$ and $E|X_n|^2 \to 1$.
(b) Give an example of random variables X_n such that $E|X_n| \to 0$ and $E|X_n|^2 \to \infty$.
(c) Give an example of a sequence of random variables X_n for which $X_n \to_p 0$ but $X_n \to_{a.s.} 0$ fails.

6. Suppose that $U \sim \text{Uniform}(0,1)$, $\alpha > 0$, and $X_n \equiv (n^\alpha / \log(n+1)) 1_{[0,1/n^\alpha]}(U)$.
(a) Show that $X_n \to_{a.s.} 0$ and $E(X_n) \to E(0) = 0$.
(b) Can you find a random variable Y with $|X_n| \le Y$ for all n with $E(Y) < \infty$ for any α?
(c) For what values of α does the uniform integrability condition

$$\limsup_{n\to\infty} E\{|X_n| 1_{[|X_n| \ge M]}\} \to 0 \qquad \text{as} \quad M \to \infty$$

hold?

7. **Optional Bonus Problem 1:** (a) Lehmann and Casella, #3.5, page 64.
(b) Lehmann and Casella, #3.6, page 64.
(c) Lehmann and Casella, #3.7, page 64.

8. **Optional Bonus Problem 2:** Suppose that $X \sim F$ on $R^+ \equiv [0,\infty)$, $Y \sim G$ on R^+, and X and Y are independent random variables. Let $Z = \min\{X, Y\} = X \wedge Y$ and $\Delta = 1\{X \le Y\}$. (This is *right-censored data*: if we view X as a survival time, and Y as a censoring time, then $Z = X$ when $X \le Y$, but $Z = Y$ when $X > Y$.)
(a) Find the joint distribution of (Z, Δ) by computing the two sub-distribution functions $F_{uc}(z) \equiv P(Z \le z, \Delta = 1)$ and $F_c(t) \equiv P(Z \le z, \Delta = 0)$.
(b) If $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$, show that Z and Δ are independent.