Statistics 581, Problem Set 10

Wellner; 11/30/2005

Reading: Chapter 4, Sections 1-4;
Ferguson, ACLST, Chapter 20, pages 133-139; Chapter 22, pages 144-150;
Lehmann and Casella, Chapter 6, especially section 6.5, pages 461-468.
Due: Wednesday, December 7, 2005.

1. **Continuation of problem 5, problem set 9:** Consider the Weibull family of example 3.2.5: $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ with $\Theta \subset R^{+2}$ given by the (Lebesgue) densities

$$p_\theta(x) = \frac{\beta}{\alpha}(\frac{x}{\alpha})^{\beta-1}\exp(-(\frac{x}{\alpha})^\beta)1_{[0,\infty)}(x)$$

where $\theta \equiv (\alpha, \beta) \in (0,\infty) \times (0,\infty) \subset R^2$. Suppose that $X, X_1, \ldots, X_n$ are i.i.d. with density function p_θ.
A. Does a maximum likelihood estimate of $\hat{\theta} = (\hat{\alpha}, \hat{\beta})$ exist? Is it unique? (See e.g. Lehmann and Casella, example 6.1, page 468.)
B. Compute an approximate (one - step) maximum likelihood estimate $\check{\theta}$ of θ using the method of moment estimators $\overline{\theta}_n$ as the preliminary estimators based on the following data (with $n = 19$):

0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50,
7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89 .

[These are failure times in minutes for an insulating fluid between two electrodes subject to a voltage of 34 kV. – from Nelson, *Applied Life Data Analysis*, page 105.]
C. Compute the maximum likelihood estimator $\hat{\theta}_n$, and compare it with the one step estimator computed in B.

2. A. Ferguson, ACLST, page 139, problem 3.
B. What if Ferguson's density $f(x|\theta)$ with $\theta \in (0,1)$ is replaced by

$$f(x|\gamma,\eta) = \{(1-\gamma)e^{-x} + \gamma\eta^2 x\exp(-\eta x)\}1_{[0,\infty)}(x)$$

with $\gamma \in (0,1)$ and $\eta > 0$?

3. Ferguson, ACLST, page 149, problem 2 modified as follows:
(a) Find the LR test statistic of the null hypothesis $H_0 : \mu = c\theta$ for any fixed number $c > 0$, and find the asymptotic distribution of the LR statistic under H_0.
(b) Does the theory of our chapter 4 (or Ferguson's chapter 22) apply directly?
(c) Does the local asymptotic power of your test depend on c?

4. Suppose that we want to model the survival of twins with a common genetic defect, but with one of the two twins receiving some treatment. Let X represent the survival time of the untreated twin and let Y represent the survival time of the treated twin. One (overly simple) preliminary model might be to assume that X and Y are independent with Exponential(η) and Exponential($\nu\eta$) distributions, respectively:

$$f_{\nu,\eta}(x,y) = \eta e^{-\eta x}\eta\nu e^{-\eta\nu y}1_{(0,\infty)}(x)1_{(0,\infty)}(y)$$

A. One crude approach to estimation in this problem is to reduce the data to $W = X/Y$, the maximal invariant for the group of scale changes $g(x, y) = (cx, cy)$ with $c > 0$. Find the distribution of W, and compute the Cramér-Rao lower bound for unbiased estimates of ν based on $W_1, \ldots, W_n$ with $W_i = X_i/Y_i$ and (X_i, Y_i) i.i.d. as (X, Y).
B. Find the information bound for estimation of ν based on observation of (X, Y) pairs when η is known and unknown.
C. Compare the bounds you computed in A and B and discuss the pros and cons of reducing to estimation based on the ratio $W = X/Y$.
D. Find the MLE $\hat{\nu}_n$ of ν based on (X_i, Y_i), $i = 1, \ldots, n$, and the MLE $\hat{\nu}_n^{(r)}$ of ν based on the reduced data $W_1, \ldots, W_n$. What are the limiting distributions of $\sqrt{n}(\hat{\nu}_n - \nu)$ and $\sqrt{n}(\hat{\nu}_n^{(r)} - \nu)$?

5. This is a continuation of the preceding problem. A more realistic model involves assuming that the common parameter η for the two twins varies across sets of twins. There are several different ways of modeling this: one approach involves supposing that each pair of twins observed (X_i, Y_i) has its own fixed parameter η_i, $i = 1, \ldots, n$. In this model we observe (X_i, Y_i) independent with densities f_{θ,η_i} for $i = 1, \ldots, n$; i.e.

$$f_{\theta,\eta_i}(x_i, y_i) = \eta_i e^{-\eta_i x_i} \eta_i \nu e^{-\eta_i \nu y_i} 1_{(0,\infty)}(x_i) 1_{(0,\infty)}(y_i) \,. \tag{0.1}$$

This is sometimes called a "functional model" (or model with incidental nuisance parameters).

Another approach is to assume that $\eta \equiv Z$ has a distribution, and that our observations are from the mixture distribution. Assuming (for simplicity) that $Z = \eta \sim \text{Gamma}(a, b)$ with density $g_{a,b}(\eta)$, it follows that the (marginal) distribution of (X, Y) is

$$\begin{aligned} p_{\nu,a,b}(x, y) &= \int_0^\infty f_{\nu,z}(x, y) g_{a,b}(z) dz \\ &= \frac{\nu}{b^2} \left(\frac{b}{b + x + \nu y} \right)^{a+2} \frac{\Gamma(a + 2)}{\Gamma(a)} \,. \end{aligned} \tag{0.2}$$

This is sometimes called a "structural model" (or mixture model).
(a) Find the information for ν in the functional model.
(b) Find the information for ν in the structural model.
(c) Compare the information bounds you computed in (a) and (b). When is the information for ν in the functional model larger than the information for ν in the structural model?
(d) Find the MLEs of ν in the functional model (call it $\hat{\nu}_n^f$) and in the structural model (call it $\hat{\nu}_n^s$). Are they both consistent estimators of ν?

6. **Optional bonus problem 1.**
Ferguson, ACLST, page 150, problem 3. Does the theory in our chapter 4 (or Ferguson's chapter 22) apply directly?

7. **Optional bonus problem 2:** Lehmann and Casella, problem 6.9, page 509.