Statistics 581, Problem Set 1

Wellner; 9/28/2005

Reading: Lehmann & Casella, TPE, pages 1 - 32; skim Chapter 0 handout; read Chapter 1 handout.
Due: Wednesday, October 5, 2005.

1. Let X and Y be i.i.d. Uniform$(0,1)$ random variables Define $U = X+Y$, $V = \min(X,Y) = X \wedge Y$.

   (i) What is the range of (U,V)?

   (ii) Find the joint density function $f_{U,V}(u,v)$ of the pair (U,V). Are U and V independent?

2. Ferguson, ACILST, #2, page 6.

3. (Continuation of the previous problem). Now suppose that $U \sim$ Uniform$(0,1)$ and for each $n \geq 1$ define $V_n \equiv \sum_{j=1}^n (j/n) 1_{((j-1)/n, j/n]}(U)$.
   (a) Show that $V_n \stackrel{d}{=} X_n$ where X_n is as in problem 2.
   (b) Show that $V_n \rightarrow_p U$.

4. Ferguson, ACILST, #6, page 7. (This is known as the Polya-Cantelli lemma; see Chapter 2, Propostion 2.11, page 10.)

5. Suppose that F is the distribution function of random variables X, Y with $X \sim$ Uniform$(0,1)$ marginally and $Y \sim$ Uniform$(0,1)$ marginally. Thus $F(x,y) = P(X \leq x, Y \leq y)$ satisfies

$$F(x,1) = x, \quad 0 \leq x \leq 1, \qquad \text{and} \qquad F(1,y) = y, \quad 0 \leq y \leq 1.$$

   (a) Show that

$$F(x,y) \leq x \wedge y \equiv F_U(x,y)$$

   for all $0 \leq x \leq 1$, $0 \leq y \leq 1$. Here $x \wedge y \equiv x$ if $x \leq y$, y if $y \leq x$.
   (b) Show that

$$F(x,y) \geq (x+y-1)^+ \equiv F_L(x,y)$$

   for all $0 \leq x \leq 1$, $0 \leq y \leq 1$. Here $z^+ = z 1_{[0,\infty)}(z)$.
   (c) Show that F_U is the distribution function of (X,X) where $X \sim$

Uniform(0, 1). Show that F_L is the distribution function of $(X, 1 - X)$ where $X \sim$ Uniform(0, 1).
(d) The distribution functions F_U and F_L are called the Fréchet bounds. Show that F_L and F_U are singular with respect to Lebesgue measure λ_2 on $[0, 1]^2$; i.e. show that the corresponding probability measures P_L and P_U satisfy

$$P((X, Y) \in A) = 1, \qquad \lambda_2(A) = 0$$

and

$$P((X, Y) \in A^c) = 0, \qquad \lambda_2(A^c) = 1$$

for some set A (which will be different for P_L and P_U). This implies that F_L and F_U do not have densities with respect to Lebesgue measure on $[0, 1]^2$. (See Chapter 0, Section 3, especially Definition 3.1 and Theorem 3.1.)

6. (a) Lehmann and Casella, TPE, problem 1.2, page 62.
   (b) Lehmann and Casella, TPE, problem 1.3, page 62.