**Statistics 581, Problem Set 1 Solutions**

Wellner; 10/10/01

1. Let $X$ and $Y$ be i.i.d. Uniform$(0,1)$ random variables Define $U = X+Y$, $V = \max(X,Y) = X \vee Y$.

   **(i)** What is the range of $(U,V)$?

   **(ii)** Find the joint density function $f_{U,V}(u,v)$ of the pair $(U,V)$. Are $U$ and $V$ independent?

**Solution:** (i) The range of $(X,Y)$ is
$A = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$. The range of $(U,V)$ is

$$B = \{(u,v) : 0 \le u \le 1, u/2 \le v \le u\} \cup \{(u,v) : 1 < u \le 2, u/2 \le v \le 1\}.$$

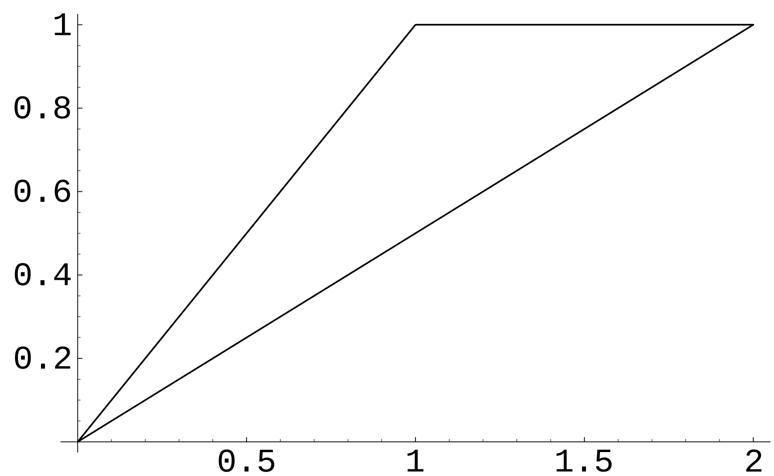


Figure 1: Range of $U, V$ .

(ii) First solution - via Jacobians: The transformation $(X,Y) \to (U,V)$ is 2-1 and onto from $A$ to $B$. On the set $x<y$, its inverse is given by $X=U-V$, $Y=V$; on the set $x>y$, its inverse is given by $X=V$, $Y=U-V$. These mappings are continuously differentiable on $B^* \equiv B \setminus \{(u,v): v=u/2\} = B \setminus$ a null set. On $B^*$ the Jacobian of the transformations are

$$\det\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = 1 \qquad \text{if } x<y, \qquad \det\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = -1 \qquad \text{if } x>y. \tag{1}$$

Thus by the usual transformation of densities formula, the joint density of $(U,V)$ is obtained from $f_{X,Y}(x,y) = 1_{[0,1]}(x)1_{[0,1]}(y)$ as follows:

$$\begin{aligned} f_{U,V}(u,v) &= f_{X,Y}(x(u,v),y(u,v))|\det\frac{\partial(x,y)}{\partial(u,v)}|1_{[x(u,v)<y(u,v)]} \\ &\quad + f_{X,Y}(x(u,v),y(u,v))|\det\frac{\partial(x,y)}{\partial(u,v)}|1_{[x(u,v)>y(u,v)]} \\ &= (1_{[0,1]}(u-v)1_{[0,1]}(v)\cdot 1 + 1_{[0,1]}(v)1_{[0,1]}(u-v)\cdot 1)1_{[v>u/2]} \\ &= 2\cdot 1_B(u,v)\,. \end{aligned}$$

Thus the joint density of $(U,V)$ is uniform on $B$. The random variables $U$ and $V$ are clearly *not* independent since the range of $(U,V)$ is not a product set in $R^2$; moreover, the joint density of $(U,V)$ does not factor into the product of its marginal densities. [The marginal densities are given by

$$f_U(u) = \int f_{U,V}(u,v)dv = \begin{cases} \int_{u/2}^{u} 2dv = u, & u\in[0,1] \\ \int_{u/2}^{1} 2dv = 2-u, & u\in(1,2] \end{cases}$$

and

$$f_V(v) = \int f_{U,V}(u,v)du = \int_v^{2v} 2du = 2v1_{[0,1]}(v).]$$

Second solution by direction calculation of the joint distribution function: Note that we can write

$$\begin{aligned} &P(U\le u, V\le v) \\ &= P(X+Y\le u, X\vee Y\le v) = P(X+Y\le u, X\le v, Y\le v) \\ &= \begin{cases} v^2 & \text{if } 0<v<1,\ 2v\le u\,, \\ v^2 - \frac{1}{2}(2v-u)^2 & \text{if } v<u<2v\,, \\ \frac{1}{2}u^2 & \text{if } 0<u<v\,. \end{cases} \end{aligned}$$

(This is easy by pictures!) Computing $(\partial^2/\partial u \partial v)P(U \le u, V \le v)$ on each of these pieces separately again yields $f_{U,V}(u,v) = 21_B(u,v)$. Also note that the marginal distribution functions of $U$ and $V$ are given by $F_U(u) = (1/2)u^2 1_{[0,1)}(u) + \{1 - \frac{1}{2}(2-u)^2\}1_{[1,2)}(u)$ on $0 \le u < 2$ and $F_V(v) = v^2$ for $0 < v < 1$.

2. Prove part (ii) of Proposition 1.1: There exists a minimal field, $\sigma-$field, and monotone class generated by any class of subsets of $\Omega$.
**Solution:** Let $\mathcal{C}$ be a class of subsets of $\Omega$. By part (i) of Proposition 1.1, the intersection of any family of fields, $\sigma-$fields, or monotone classes, are again fields, $\sigma-$fields, or monotone classes, respectively. Consider the collection $\{\mathcal{A}_\lambda\}$, of all fields containing $\mathcal{C}$. This collection is non-empty since it contains the field $2^\Omega$ containing all subsets of $\Omega$. Thus by Proposition 1.1(i) $\cap_\lambda \mathcal{A}_\lambda$ is again a field. This is clearly the minimal field containing $\mathcal{C}$ since $\cap_\lambda \mathcal{A}_\lambda \subset \mathcal{A}_{\lambda'}$ for each particular field $\mathcal{A}_{\lambda'}$ containing $\mathcal{C}$. The proofs for $\sigma-$fields and monotone classes is exactly the same.

3. (a) Suppose that $\{\mathcal{A}_n\}$ is an increasing sequence of fields, i.e. $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ for all $n \ge 1$. Show that $\cup_{n=1}^\infty \mathcal{A}_n$ is a field. (b) Suppose that the $\mathcal{A}_n$ of (a) are $\sigma-$ fields. Show by constructing a counterexample that $\cup_{n=1}^\infty \mathcal{A}_n$ need not be a $\sigma-$field.
**Solution:** (a) Let $\mathcal{A} = \cup_{n=1}^\infty \mathcal{A}_n$.
(i) If $A \in \mathcal{A}$, then $A \in \mathcal{A}_n$ for some $n$, so $A^c \in \mathcal{A}_n$ (since $\mathcal{A}_n$ is a field), which implies $A^c \in \mathcal{A}$.
(ii) If $A, B \in \mathcal{A}$, then $A \in \mathcal{A}_m$, $B \in \mathcal{A}_n$ for some $m, n$. Suppose that $m \le n$. Then $A \in \mathcal{A}_n$ since $\mathcal{A}_m \subset \mathcal{A}_{m+1} \subset \ldots \subset \mathcal{A}_n$. Hence $A \cup B \in \mathcal{A}_n$ (since $\mathcal{A}_n$ is a field). Hence $A \cup B \in \cup_{n=1}^\infty \mathcal{A}_n = \mathcal{A}$.
By (i) and (ii) $\mathcal{A}$ is a field.
(b) Let $\mathcal{A}_1 = \sigma(\{\emptyset, [0,1]\})$, $\mathcal{A}_2 = \sigma(\mathcal{A}_1 \cup \{[0,1/2]\})$, $\mathcal{A}_3 = \sigma(\mathcal{A}_2 \cup \{[0, 1 - 1/3]\})$, ..., $\mathcal{A}_n = \sigma(\mathcal{A}_{n-1} \cup \{[0, 1-1/n]\})$, .... Then each $\mathcal{A}_n$ is a finite $\sigma-$field with $\sharp(\mathcal{A}_n) = 2^n$; clearly $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ by construction. Consider the sets $A_n = [0, 1-1/n] \in \mathcal{A}_n$ for $n \ge 2$. Then $A = \cup_{n=2}^\infty A_n = [0,1) \notin \cup_{n=1}^\infty \mathcal{A}_n$. [If $A = [0,1) \in \cup_{n=1}^\infty \mathcal{A}_n$, then $A \in \mathcal{A}_{n_0}$ for some $n_0$, but this fails. All the sets in $\cup_{n=1}^\infty \mathcal{A}_n$ are closed on the right.]

4. Let $\mu$ and $\nu$ be Lebesgue-Stieltjes measures on $(R, \mathcal{B})$ with corresponding generalized d.f.'s $F$ and $G$. Show that:
(a) $\int_{(a,b]} (F(y) - F(a)) dG(y) = (\mu \times \nu)(\{(x,y) : a < x \le y \le b\})$.

(b) $\int_{(a,b]} F(y)dG(y)+\int_{(a,b]} G(y)dF(y) = F(b)G(b)-F(a)G(a)+\sum_{x\in(a,b]} \mu(\{x\})\nu(\{x\})$.
To see that the second term is needed, let $F(x) = G(x) = 1_{[0,\infty)}(x)$ and $a < 0 < b$.
(c) If $F = G$ is continuous, then $\int_{(a,b]} F(y)dF(y) = (F^2(b)-F^2(a))/2$.
[Hint: use Fubini's theorem.]
**Solution:** (a) Now $F(y) - F(a) = \mu((a, y]) = \int_{(a,y]} d\mu(x)$ so, by using the Tonelli part of the Fubini-Tonelli theorem,

$$\begin{aligned}
\int_{(a,b]} (F(y) - F(a))dG(y) &= \int_{(a,b]} \left(\int_{(a,y]} d\mu(x)\right) d\nu(y) \\
&= \int_{(a,b]} \int_{(a,b]} 1_{[x\le y]} d\mu(x) d\nu(y) \\
&= (\mu\times\nu)(\{(x, y) : a < x \le y \le b\}) \\
&\equiv (\mu\times\nu)(B)
\end{aligned}$$

where the set $B \equiv \{(x, y) : a < x \le y \le b\}$.
(b) By using (a) we compute

$$\begin{aligned}
\int_{(a,b]} F(y)dG(y) &= \int_{(a,b]} (F(y) - F(a) + F(a))dG(y) \\
&= (\mu\times\nu)(B) + F(a)(G(b) - G(a)) .
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{(a,b]} G(y)dF(y) &= \int_{(a,b]} (G(y) - G(a) + G(a))dF(y) \\
&= (\mu\times\nu)(A) + G(a)(F(b) - F(a))
\end{aligned}$$

where $A \equiv \{(x, y) : a < y \le x \le b\}$. Note that

$$A \cap B = \{(x, x) : a < x \le b\} \equiv D .$$

Thus by adding these two expressions we find that

$$\begin{aligned}
&\int_{(a,b]} F(y)dG(y) + \int_{(a,b]} G(y)dF(y) \\
&= (\mu\times\nu)(B) + F(a)(G(b) - G(a)) \\
&\qquad + (\mu\times\nu)(A) + G(a)(F(b) - F(a))
\end{aligned}$$

$$\begin{aligned}
&= (\mu\times\nu)((a,b]\times(a,b]) + (\mu\times\nu)(D) \\
&\qquad + F(a)(G(b)-G(a)) + G(a)(F(b)-F(a)) \\
&= (F(b)-F(a))(G(b)-G(a)) + F(a)(G(b)-G(a)) + G(a)(F(b)-F(a)) \\
&\qquad + (\mu\times\nu)(D) \\
&= F(b)G(b) - F(a)G(a) + (\mu\times\nu)(D) \\
&= F(b)G(b) - F(a)G(a) + \sum_{x\in(a,b]} \mu(\{x\})\nu(\{x\})
\end{aligned}$$

where the last line follows since the Lebesgue-Stieltjes measures $\mu$ and $\nu$ can have at most a countable number of discontinuity points in the corresponding generalized d.f.'s $F$ and $G$ in any bounded set.
Note that for the example given with $F(x) = G(x) = 1_{[0,\infty)}(x)$ and $a < 0 < b$, the left side of the formula is $\int_{(a,b]} F(y)dG(y) + \int_{(a,b]} G(y)dF(y) = 1 + 1 = 2$, while the right side is

$$F(b)G(b) - F(a)G(a) + \sum_{x\in(a,b]} \mu(\{x\})\nu(\{y\}) = 1 - 0 + \mu(0)\nu(\{0\}) = 1 + 1 = 2\,.$$

(c) If $F = G$ is continuous, then from (b) it follows that

$$2\int_{(a,b]} FdF = \int_{(a,b]} FdG + \int_{(a,b]} GdF = F(b)^2 - F(a)^2 + 0\,.$$

5. Let $\mathcal{X} = (0,1)$, $\mathcal{Y} = (0,1)$, both equipped with the Borel sets and Lebesgue measure. Let

$$g(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2} \qquad \text{for} \quad (x,y) \in (0,1)\times(0,1)\,.$$

Show that:
(a) $\int_0^1 (\int_0^1 g(x,y)dy)dx = \pi/4$.
(b) $\int_0^1 (\int_0^1 g(x,y)dx)dy = -\pi/4$.
(c) Why does Fubini's theorem fail here?
**Solution:** (a) It is easily seen that

$$\int_0^1 g(x,y)dy = \frac{y}{x^2+y^2}\Big|_{y=0}^{1} = \left(\frac{1}{1+x^2} - 0\right) = \frac{1}{1+x^2}\,,$$

and hence

$$\int_0^1 (\int_0^1 g(x,y)dy)dx = \int_0^1 \frac{1}{1+x^2}dx = \arctan(x)|_0^1 = \pi/4\,.$$

(b) Similarly,

$$\int_0^1 g(x,y)dx = -\frac{x}{x^2+y^2}\Big|_{x=0}^1 = -\left(\frac{1}{1+y^2} - 0\right) = \frac{-1}{1+y^2}\,,$$

and hence

$$\int_0^1 (\int_0^1 g(x,y)dy)dx = -\int_0^1 \frac{1}{1+y^2}dy = -\arctan(y)|_0^1 = -\pi/4\,.$$

(c) It is clear that the hypothesis $g \in L_1(\lambda \times \lambda)$ fails in this example; if it held, the two iterated integrals in (a) and (b) must be the same by the Fubini part of the Fubini-Tonelli theorem. To see that $g \notin L_1(\mu \times \nu)$ directly, note that the function $g$ is non-negative on the set $0 \leq y \leq x \leq 1$, and non-positive on the set $0 \leq x \leq y \leq 1$. Thus

$$\int_0^1 \int_0^1 |g(x,y)|dxdy = \int\int_{\{0\leq y\leq x\leq 1\}} g(x,y)dxdy + \int\int_{\{0\leq x\leq y\leq 1\}} -g(x,y)dxdy\,,$$

where, by the Tonelli part of the Fubini-Tonelli theorem

$$\begin{aligned}
\int\int_{\{0\leq y\leq x\leq 1\}} g(x,y)dxdy &= \int_0^1 \left(\int_y^1 g(x,y)dx\right) dy \\
&= \int_0^1 \left(\frac{-x}{x^2+y^2}\Big|_{x=y}^1 dy\right. \\
&= \int_0^1 \left(\frac{-1}{1+y^2} - \frac{-y}{2y^2}\right) dy \\
&= \int_0^1 \left(\frac{1}{2y} - \frac{1}{1+y^2}\right) dy \\
&= \infty - \pi/4 = \infty\,.
\end{aligned}$$

By symmetry it follows that

$$\int\int_{\{0\leq x\leq y\leq 1\}} -g(x,y)dxdy = \infty\,,$$

and hence we have $\int\int |g(x,y)|dxdy = \infty$. Thus $g \notin L_1(\lambda \times \lambda)$ on $[0,1] \times [0,1]$.

6. Lehmann and Casella, TPE, problem 1.4, page 62: Let $X$ and $Y$ have common expectation $\theta$, variances $\sigma^2$ and $\tau^2$, and correlation coefficient $\rho$. Determine the conditions on $\sigma$, $\tau$, and $\rho$ under which:
(a) $Var(X) < Var[(X+Y)/2]$;
(b) The value of $\alpha$ that minimizes $Var(\alpha X + (1-\alpha)Y)$ is negative.
**Solution:** (a) First note that

$$Var((X+Y)/2) = \frac{1}{4}(\sigma^2 + \tau^2 + 2\rho\sigma\tau) .$$

Thus the inequality in question holds if

$$\sigma^2 + \tau^2 + 2\rho\sigma\tau > 4\sigma^2 ,$$

or equivalently if

$$3\sigma^2 - 2\rho\tau\sigma - \tau^2 < 0 .$$

Since the left side, viewed as a function of $\sigma$ has zeros at

$$\begin{aligned} \sigma_0 &= \frac{2\rho\tau \pm \sqrt{4\rho^2\tau^2 - 4(3)(-\tau^2)}}{6} \\ &= \frac{\tau}{3}\left(\rho \pm \sqrt{\rho^2+3}\right) , \end{aligned}$$

it follows that the inequality in (a) holds for given $\tau$ and $\rho$ if

$$0 \le \sigma < \frac{\tau}{3}(\rho + \sqrt{\rho^2+3}) .$$

When $\rho = -1$, this requires $\sigma < \tau/3$; when $\rho = 0$, we need $\sigma < \tau/\sqrt{3}$; when $\rho = 1$, we only need $\sigma < \tau$.
(b) First note that

$$\begin{aligned} f(\alpha) &\equiv Var(\alpha X + (1-\alpha)Y) = Var(\alpha(X-Y) + Y) \\ &= \alpha^2 Var(X-Y) + 2\alpha Cov(X-Y, Y) + Var(Y) . \end{aligned}$$

Hence

$$f'(\alpha) = 2\alpha Var(X-Y) + 2Cov(X-Y, Y)$$

and $f''(\alpha) = 2Var(X-Y)$. It follows that $f$ is minimized as a function of $\alpha$ by

$$\alpha_{min} = \frac{-Cov(X-Y,Y)}{Var(X-Y)} = \frac{Cov(Y-X,Y)}{Var(X-Y)} = \frac{\tau^2 - \rho\sigma\tau}{\sigma^2+\tau^2-2\rho\sigma\tau} ,$$

and this is negative if $\tau^2 - \rho\sigma\tau < 0$, or equivalently if $\tau < \rho\sigma$. For example, if $\rho = 1/2$, $\sigma = 1$ and $\tau = 1/4$, then the minimizing value of $\alpha$ is

$$\alpha_{min} = \frac{1/16 - (1/2)(1)(1/4)}{1 + 1/16 - 2(1/2)(1)(1/4)} = \frac{-1/16}{13/16} = \frac{-1}{13} .$$

Note that if $\rho = 0$, then $\alpha_{min} = \tau^2/(\sigma^2 + \tau^2)$; if the variances are equal, this further simplifies to $\alpha_{min} = 1/2$.

7. Lehmann and Casella, TPE, problem 1.10, page 62. Show that for real numbers $x_1, \ldots, x_n$ in the domain of a monotone real-valued functions $h$, the function

$$H(a) \equiv \frac{1}{n}\sum_{i=1}^{n}(h(x_i) - h(a))^2$$

is minimized by that value of $a$, say $a_{min}$, given by

$$a_{min} = h^{-1}\left(\frac{1}{n}\sum_{1}^{n} h(x_i)\right) .$$

Find particular functions $h$ so that this yields the arithmetic, harmonic, and geometric means of the $x_i$'s.

**Solution:** First note that that $H(a)$ is minimized by any value of $a$, say $a_{min}$, satisfying

$$h(a_{min}) = \frac{1}{n}\sum_{1}^{n} h(x_i) .$$

This follows by noting that

$$H(a) = H(a_{min}) + (h(a_{min}) - h(a))^2 .$$

Since we have assumned that $h$ is (strictly) monotone, it follows easily that

$$a_{min} = h^{-1}\left(\frac{1}{n}\sum_{1}^{n} h(x_i)\right) .$$

When $h(x) = x$,

$$a_{min} = \frac{1}{n}\sum_{1}^{n}(x_i) = \text{the arithmetic mean} .$$

When $h(x) = 1/x$,

$$a_{min} = \frac{1}{\frac{1}{n}\sum_1^n \frac{1}{x_i}} = \text{the harmonic mean}\,.$$

When $h(x) = \log(x)$,

$$\begin{aligned} a_{min} &= \exp\left(\frac{1}{n}\sum_1^n \log(x_i)\right) = \exp\left(\frac{1}{n}\log(\prod_1^n x_i)\right) \\ &= \exp\left(\log\left((\prod_1^n x_i)^{1/n}\right)\right) = \left(\prod_1^n x_i\right)^{1/n} \\ &= \text{the geometric mean}\,. \end{aligned}$$