Statistics 581, Problem Set 9

Wellner; 11/28/2001

Reading: Chapter 4, Sections 1-2;
Ferguson, ACLST, Chapter 20, pages 133-139; Chapter 22, pages 144-150;
Lehmann and Casella, Chapter 6, especially section 6.5, pages 461-468.
Due: Wednesday, December 5, 2001.

1. (a) Lehmann and Casella, problem 6.3.1, page 501.
   (b) Lehmann and Casella, problem 6.3.2, page 501.
   (c) Lehamnn and Casella, problem 6.3.4, page 501.
   (d) Lehmann and Casella, problem 6.3.18, page 502. [**Note:** It seems to me that 3.15(b) should be 3.15(c) since $C(0,a)$ is a *scale family*).]

2. Suppose that $(Y|Z) \sim$ Poisson$(\lambda e^{\gamma Z})$, and $Z \sim Bernoulli(\eta)$, and $\theta = (\lambda, \gamma, \eta)$. Let $X = (Y, Z)$, and suppose that we observe $X_1, \ldots, X_n$ i.i.d. as X.
   (a) Find the score equations for estimation of θ.
   (b) Give conditions on the data $X_1, \ldots, X_n = (Y_1, Z_1), \ldots, (Y_n, Z_n)$ guaranteeing that the score equations have a unique solution which maximizes the likelihood. Call the resulting estimators $\widehat{\theta}_n = (\widehat{\lambda}_n, \widehat{\gamma}_n, \widehat{\eta}_n)$.
   (c) What does theorem 4.1.5 (Chapter 4, page 4), say about the asymptotic distribution of $\sqrt{n}(\widehat{\theta} - \theta_0)$ when the distribution of the data is given by P_{θ_0}.
   (d) Suppose that $\theta_1 \neq \theta_0$ is the "true" value of the parameter θ, and we consider the likelihood ratio $L_n(\theta_1)/L_n(\theta_0)$ where $L_n(\theta) \equiv \prod_{i=1}^n p_\theta(X_i)$. Show that $n^{-1} \log(L_n(\theta_1)/L_n(\theta_0) \rightarrow_p$ some constant, and identify the constant explicitly in terms of θ_1, θ_0.

3. For the same set-up as in problem 1, consider taking a "profile likelihood" approach to the estimation of γ as follows:
   (a) Let $l_n(\theta) = l_n(\gamma, \lambda, \eta)$: consider first maximizing this as a function of λ and η for each fixed value of γ to find
   $$(\widehat{\lambda}(\gamma), \widehat{\eta}(\gamma)) \equiv \text{argmax}_{(\lambda,\eta)} l_n(\lambda, \gamma, \eta)\,.$$
   Compute the maximizer $(\widehat{\lambda}(\gamma), \widehat{\eta}(\gamma))$ as explicitly as possible, and then form the "profile log-likelihood" $l_n^{profile}(\gamma)$ defined by
   $$l_n^{profile}(\gamma) \equiv l_n(\widehat{\lambda}(\gamma), \gamma, \widehat{\eta}(\gamma))\,.$$
   (b) Now maximize $l_n^{profile}(\gamma)$ with respect to γ. Find the resulting "profile likelihood" score equation for γ.
   (c) Does the equation you derived in (b) follow from the original score equations?
   (d) Does the "profile score function" which appears in (b) correspond to or relate to the efficient score for γ in any way?

4. Consider the Weibull family of example 3.2.5: $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ with $\Theta \subset R^{+2}$ given by the (Lebesgue) densities
   $$p_\theta(x) = \frac{\beta}{\alpha}(\frac{x}{\alpha})^{\beta-1} \exp(-(\frac{x}{\alpha})^\beta) 1_{[0,\infty)}(x)$$

where $\theta \equiv (\alpha, \beta) \in (0, \infty) \times (0, \infty) \subset R^2$. Suppose that $X, X_1, \ldots, X_n$ are i.i.d. with density function p_θ.
A. If $X \sim P_\theta \in \mathcal{P}$, show that the distributions of $\log X$ form a location and scale family from a Gumbel (extreme value) density on R.
B. Use the result of A to construct method of moments estimators or quantile based estimators $\overline{\theta}_n$ of $\theta = (\alpha, \beta)$.
C. Show that the method of moments or quantile estimators $\overline{\theta}_n$ of θ are asymptotically normal, and find the asymptotic distribution; i.e. show that

$$\sqrt{n}(\overline{\theta}_n - \theta) \rightarrow_d N_2(0, \Sigma) \qquad \text{for some} \qquad \Sigma.$$

D. Does a maximum likelihood estimate of $\hat{\theta} = (\hat{\alpha}, \hat{\beta})$ exist? Is it unique?
E. Compute an approximate (one - step) maximum likelihood estimate $\check{\theta}$ of θ using the method of moment estimators $\overline{\theta}_n$ as the preliminary estimators based on the following data (with $n = 19$):

0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50,
7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89 .

[These are failure times in minutes for an insulating fluid between two electrodes subject to a voltage of 34 kV. – from Nelson, *Applied Life Data Analysis*, page 105.]
F. Compute the maximum likelihood estimator $\hat{\theta}_n$, and compare it with the one step estimator computed in E.

5. **Optional bonus problem:** Lehmann and Casella, TPE, problem 6.3.22, page 503, reworded as follows. (In other words, prove (vi) of theorem 1.5, page 5, chapter 4 notes). Suppose that $X_1, \ldots, X_n$ are i.i.d. with density p_θ, $\theta \in \Theta \subset R^k$, satisfying the hypotheses of theorem 4.1, page 429 (the Cramér conditions given in (A) - (D) on page 429). Show that the following Local Asymptotic Normality (LAN) result holds for the (local) log- likelihood ratios: with

$$L_n(\theta) \equiv \log(\prod_{i=1}^{n} p_\theta(X_i)) = \sum_{i=1}^{n} \log p_\theta(X_i),$$

for a fixed $\theta_0 \in \Theta$,

$$\begin{aligned} L_n(\theta_0 + n^{-1/2}\underline{t}) - L_n(\theta_0) &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \underline{t}^T \dot{\underline{l}}_\theta(X_i) - \frac{1}{2}\underline{t}^T I(\theta_0)\underline{t} + o_p(1) \\ &\rightarrow_d \; N(0, t^T I(\theta_0)t) - \frac{1}{2}t^T I(\theta_0)t =_d N(-\frac{1}{2}\sigma^2 \sigma^2) \end{aligned}$$

under P_{θ_0} where $\sigma^2 \equiv \underline{t}^T I(\theta_0)\underline{t}$.