Statistics 581, Problem Set 8

Wellner; 11/21/2001

Reading: Chapter 3, Section 2;
Ferguson, ACLST, Chapter 19, pages 126-132, Chapter 20, pages 133-134;
Lehmann and Casella, pages 113-129, and 439- 443;
begin reading Chapter 4 (to be handed out on Wednesday 11/21).
Due: Wednesday, November 28, 2001.

1. Suppose that $\theta = (\theta_1, \theta_2 \in \Theta \subset R^k$ where $\theta_1 \in R$ and $\theta_2 \in R^{k-1}$. Show that:
   A. $l_1^* = \dot{l}_1 - I_{12} I_{22}^{-1} \dot{l}_2$ is orthogonal to $[\dot{l}_2] \equiv \{a' \dot{l}_2 : a \in R^{k-1}\}$ in $L_2(P_\theta)$.
   B. $I_{11 \cdot 2} = \inf_{c \in R^{k-1}} E_\theta (\dot{l}_1 - c' \dot{l}_2)^2$ and that the minimum is achieved when $c' = I_{12} I_{22}^{-1}$. Thus
   $$I_{11 \cdot 2} = E_\theta (\dot{l}_1 - I_{12} I_{22}^{-1} \dot{l}_2)^2 = E_\theta [(l_\theta^*)^2] \, .$$
   C. Prove the formula (14) on page 16 of Chapter 3 and interpret this formula geometrically.

2. Suppose that $(Y|Z) \sim \text{Poisson}(\lambda e^{\gamma Z})$, and $Z \sim G_\eta$ on R with density g_η with respect to some dominating measure μ. You may assume that
   $$a(z) \equiv (\partial / \partial \eta) \log g_\eta(z)$$
   exists and $E\{a^2(Z)\} < \infty$. Thus Z is a "covariate" or "predictor variable", γ is a "regression parameter" which affects the intensity of the (conditionally) Poisson variable Y, and $\theta = (\lambda, \gamma, \eta)$.
   (a) Find the information matrix for θ. What does the structure of this matrix say about the effect of η being known or unknown about the estimation of λ and γ?
   (b) Find the information and information bound for γ if the parameter λ is known.
   (c) Find the efficient score function and the efficient influence function for estimation of γ when λ is known.
   (d) Find the information and information bound for γ if the parameter λ is unknown, $I_{\gamma\gamma\cdot\lambda}$.
   (e) Find the efficient score function and the efficient influence function for estimation of γ when λ is unknown.
   (f) In the case when $Z \sim \text{Bernoulli}(\eta)$, compute the ratio of the information when λ is unknown, to the information when λ is known as a function of γ and of η.

3. Information for location-scale families. Example 6.5, TPE page 126.
   A. Confirm that Lehmann's information matrix for (regular) location-scale families is correct.
   B. Verify that the off-diagonal term $I_{12} = 0$ when the location -scale family is from a density f that is symmetric about 0, and interpret this geometrically in terms of the scores for location $\mu = \theta_1$ and for scale $\sigma = \theta_2$.
   C. Produce an example of a location-scale family which is not symmetric about 0 and hence for which $I_{12} \neq 0$. Compute the informatio matrix $I(\theta)$ as explicitly as possible in this case.

4. **Optional bonus problem:** Suppose that $X \sim \text{Gamma}(\alpha, \beta)$; i.e. X has density p_θ given by

$$p_\theta(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x) 1_{(0,\infty)}(x), \quad \theta = (\alpha, \beta) \in (0, \infty) \times (0, \infty) \equiv \Theta\,.$$

Consider estimation of : A. $q_A(\theta) \equiv E_\theta X$. B. $q_B(\theta) \equiv F_\theta(x_0)$ for a fixed x_0; here $P_\theta(x) \equiv P_\theta(X \le x)$.
(i) Compute $I(\theta) = I(\alpha, \beta)$; compare Lehmann & Casella page ??
(ii) Compute $q_A(\theta)$, $q_B(\theta)$, $\dot{q}_A(\theta)$, and $\dot{q}_B(\theta)$.
(iii) Find the efficient influence functions for estimation of q_A and q_B.
(iv) Compare the efficient influence functions you find in (iii) with the influence functions ψ_A and ψ_B of the natural nonparametric estimators $\overline{X}_n$ and $\mathbb{F}_n(x_0)$ respectively; in particular, show that $\psi_A \in \dot{\mathcal{P}}$, while $\psi_B \notin \dot{\mathcal{P}}$.