Statistics 581, Problem Set 4
Wellner; 10/24/00

Reading: Lehmann & Casella, TPE, Chapter 5, pages 352 - 359. Chapter 2, sections 3 and 4; Ferguson, ACILST pages 44 - 66.
Due: Wednesday, October 31, 2000.

1. Suppose that $\underline{N}_n \sim$ Mult$_k(n, \underline{p})$ and $\widehat{\underline{p}} = \underline{N}_n/n$. Suppose that $g : R^k \to R^k$ is of the form $g(\underline{x}) = (g_1(x_1), \ldots, g_k(x_k))$ where each g_j is differentiable. Then the "transformed chi-square statistic" $C_n(g)$ is defined by
$$C_n(g) \equiv C_n(g, \underline{p}) = n \sum_{j=1}^{k} \frac{(g_j(\widehat{p}_j) - g_j(p_j))^2}{p_j \dot{g}_j(p_j)^2}.$$
(a) Show that $C_n(g) \to_d \chi^2_{k-1}$.
(b) Specialize this to the case $g_j(x_j) = x_j^{1/2}$, and show that the resulting statistic is related to the Hellinger distance between $\widehat{\underline{p}}$ and $\underline{p}$.
(c) Suppose that the "true $\underline{p}$ is $\underline{p}_n = \underline{p_0} + n^{-1/2}\underline{c}$. Thus $\underline{N}_n = \sum_{i=1}^n \underline{M}_{ni}$ where $\underline{M}_{n1}, \ldots, \underline{M}_{n1}$ are i.i.d. Mult$_k(1, \underline{p}_n)$.
Show that $C_n(g, \underline{p}_0) \to_d \chi^2_{k-1}(\delta)$ where $\delta = \sum_{j=1}^k c_j^2/p_{0j}$.
[Hint: See Ferguson pages 59 and 66.]

2. Ferguson, ACILST, problem 1, page 65.

3. Ferguson, ACILST, problem 3, page 42.

4. Suppose that $X_1, X_2, \ldots$ are i.i.d. (μ, σ^2) with $\mu_4 < \infty$. Let $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$ and $S_n^2 = (n-1)^{-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2$ be the sample mean and sample variance respectively.
(a) Show that
$$\sqrt{n}\begin{pmatrix} \overline{X}_n - \mu \\ S_n^2 - \sigma^2 \end{pmatrix} \to_d \underline{Z} \sim N_2(0, \Sigma)$$
where
$$\begin{pmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{pmatrix}.$$
(b) Suppose $\sigma > 0$. Use (a) to find the limiting distribution of the sample *signal-noise ratio* $D_n \equiv \overline{X}_n/S_n$; i.e. show that $\sqrt{n}(D_n - d) \to_d N(0, V^2)$ with $d \equiv \mu/\sigma$ and find V^2.

5. Ferguson, ACILST, problem 5, page 50.

6. **Optional bonus problem:** (a) Show that if:
(i) $X \sim N_k(\delta, \Sigma)$ where Σ is of rank $r \le k$,
(ii) Σ is a projection matrix (so $\Sigma^2 = \Sigma$).
(iii) $\Sigma\delta = \delta$,
then $X^T X \sim \chi^2_r(\delta^T\delta)$.

7. Ferguson, ACILST, problem 6, page 50