

Statistics 582, Problem Set 2

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Reading: Chapter 1, especially pages 13 - 17; start reading chapter 2; Ferguson pages 1-25.

Due: Wednesday, October 17, 2001.

1. Ferguson, ACILST, #2, page 6:
 - (a) Suppose that $X_n \sim \text{Uniform}\{1/n, 2/n, \dots, n/n\}$. Show that $X_n \rightarrow_d X \sim \text{Uniform}(0, 1)$. Does $X_n \rightarrow_p X$?
 - (b) Suppose that $Y_n = \sum_{k=1}^n (k/n) 1_{[(k-1)/n, k/n)}(U)$ where $U \sim \text{Uniform}[0, 1]$. Show that $Y_n \sim \text{Uniform}\{1/n, 2/n, \dots, n/n\}$, and $Y_n \rightarrow_p U$.

2. Ferguson, ACILST, #4, page 6:

Give an example of random variables X_n such that $E|X_n| \rightarrow 0$ and $E|X_n|^2 \rightarrow 1$.

3. Suppose that $U \sim \text{Uniform}(0, 1)$, $\alpha > 0$, and

$$X_n \equiv (n^\alpha / \log(n+1)) 1_{[0, 1/n^\alpha]}(U).$$

- (a) Show that $X_n \rightarrow_{a.s.} 0$ and $E(X_n) \rightarrow E(0) = 0$.
- (b) Can you find a random variable Y with $|X_n| \leq Y$ for all n with $E(Y) < \infty$ for any α ?
- (c) For what values of α does the uniform integrability condition

$$\limsup_{n \rightarrow \infty} E\{|X_n| 1_{\{|X_n| \geq M\}}\} \rightarrow 0 \quad \text{as } M \rightarrow \infty$$

hold?

4. Suppose that $X \sim \text{Uniform}(0, 1)$ and $Y = X^2$.
 - (a) Find the joint distribution function $F(x, y) = F_{X,Y}(x, y)$ of (X, Y) .
 - (b) Is F a continuous function?
 - (c) Is the probability measure P corresponding to F absolutely continuous with respect to Lebesgue measure μ on R^2 ?

5. (a) Lehmann and Casella, #3.5, page 64.
 (b) Lehmann and Casella, #3.6, page 64.
 (c) Lehmann and Casella, #3.7, page 64.
6. Suppose that $X \sim F$ on $R^+ \equiv [0, \infty)$, $Y \sim G$ on R^+ , and X and Y are independent random variables. Let $Z = \min\{X, Y\} = X \wedge Y$ and $\Delta = 1\{X \leq Y\}$. (This is *right-censored data*: if we view X as a survival time, and Y as a censoring time, then $Z = X$ when $X \leq Y$, but $Z = Y$ when $X > Y$.)
- (a) Find the joint distribution of (Z, Δ) .
 (b) If $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$, show that Z and Δ are independent.
 [Hint: for (a), compute $P(Z \leq z, \Delta = 1)$ and $P(Z \leq z, \Delta = 0)$.]
7. **Optional Bonus Problem 1:** Suppose that X and Y are as in the preceding problem, but that we now observe (Y, Δ) ; this is called “interval censored” or “current-status” data.
- (a) Find the joint distribution of (Y, Δ) .
 (b) Specialize the result in (a) when $X \sim \text{Exp}(\lambda)$ and $Y \sim \text{Exp}(\mu)$ as in (b) of the preceding problem.
8. **Optional Bonus Problem 2:**
- (a) If X is a random variable on a probability space (Ω, \mathcal{A}, P) , show that the distribution function $F = F_X$ of X defined by $F(x) = P(X \leq x)$ is right-continuous.
 (b) Show that the discontinuity set $D_F \equiv \{x \in R : F \text{ is discontinuous at } x\}$ of F is countable. Hint: consider the intervals $(F(x-), F(x)]$.
 (c) Define the inverse F^{-1} of F by $F^{-1}(u) \equiv \inf\{x : F(x) \geq u\}$ for $0 < u < 1$. Show that F^{-1} is left-continuous.
 (d) Show that the discontinuity set $D_{F^{-1}} = \{u \in (0, 1) : F^{-1} \text{ is discontinuous at } u\}$ is countable.