Statistics 581, Problem Set 1

Wellner; 10/03/01

Reading: Lehmann & Casella, TPE, pages 1 - 32; all of Chapter 0 handout; start reading Chapter 1 handout.
Due: Wednesday, October 10, 2001.

1. Let X and Y be i.i.d. Uniform$(0,1)$ random variables Define $U = X+Y$, $V = \max(X,Y) = X \vee Y$.

   (i) What is the range of (U,V)?

   (ii) Find the joint density function $f_{U,V}(u,v)$ of the pair (U,V). Are U and V independent?

2. Prove part (ii) of Proposition 1.1: There exists a minimal field, $\sigma-$field, and monotone class generated by any class of subsets of Ω.

3. (a) Suppose that $\{\mathcal{A}_n\}$ is an increasing sequence of fields, i.e. $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ for all $n \geq 1$. Show that $\cup_{n=1}^{\infty} \mathcal{A}_n$ is a field. (b) Suppose that the $\mathcal{A}_n$ of (a) are $\sigma-$ fields. Show by constructing a counterexample that $\cup_{n=1}^{\infty} \mathcal{A}_n$ need not be a $\sigma-$field.

4. Let μ and ν be Lebesgue-Stieltjes measures on $(R, \mathcal{B})$ with corresponding generalized d.f.'s F and G. Show that:
   (a) $\int_{(a,b]} (F(y) - F(a)) dG(y) = (\mu \times \nu)(\{(x,y) : a < x \leq y \leq b\})$.
   (b) $\int_{(a,b]} F(y) dG(y) + \int_{(a,b]} G(y) dF(y) = F(b)G(b) - F(a)G(a) + \sum_{x \in (a,b]} \mu(\{x\})\nu(\{x\})$.
   To see that the second term is needed, let $F(x) = G(x) = 1_{[0,\infty)}(x)$ and $a < 0 < b$.
   (c) If $F = G$ is continuous, then $\int_{(a,b]} F(y) dF(y) = (F^2(b) - F^2(a))/2$.
   [Hint: use Fubini's theorem.]

5. Let $\mathcal{X} = (0,1)$, $\mathcal{Y} = (0,1)$, both equipped with the Borel sets and Lebesgue measure. Let

$$g(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2} \qquad \text{for } \ (x,y) \in (0,1) \times (0,1)\,.$$

Show that:
(a) $\int_0^1 (\int_0^1 g(x,y)dy)dx = \pi/4$.
(b) $\int_0^1 (\int_0^1 g(x,y)dx)dy = -\pi/4$.
(c) Why does Fubini's theorem fail here?

6. Lehmann and Casella, TPE, problem 1.4, page 62.

7. Lehmann and Casella, TPE, problem 1.10, page 62.