**Statistics 581, Midterm Exam**

Wellner; 11/05/2001

1. (24 points) **Define** *any three* of the following terms. In each case, provide an appropriate context for your definition.
(a) A $\sigma-$field $\mathcal{A}$ of subsets of a set $\Omega$.
(b) The event $[A_n \ i.o.]$.
(a) A measurable function $X$.
(d) Convergence in probability.
(e) Convergence in distribution (of a sequence of random variables).
(f) The inverse or quantile function $F^{-1}$ of a distribution function $F$.

2. (24 points) **State** *any three* of the following results:
(a) The monotone convergence theorem.
(b) The dominated convergence theorem.
(c) Fatou's lemma.
(d) The Mann-Wald theorem.
(e) Scheffé's theorem.
(f) The Liapunov central limit theorem.

3. (30 points) A sequence of random variables $X_n$ is "bounded in probability", which we express in symbols as $X_n = O_p(1)$, if for every $\epsilon > 0$ there exist $M$ and $n_0$ such that $P(|X_n| > M) < \epsilon$ for all $n > n_0$; i.e. if
$$\lim_{M\to\infty} \limsup_{n\to\infty} P(|X_n| > M) = 0 .$$
We write $X_n = O_p(b_n)$ for a sequence of positive real numbers $b_n$ if $X_n/b_n = O_p(1)$.
(a) Show that if $X_n \to_d X$, then $X_n = O_p(1)$.
Now suppose that $X_1, X_2, X_3, \ldots$ are i.i.d. with mean $\mu$ and variance $\sigma^2$ (so $E(X^2) < \infty$. Let $S_n = X_1 + \ldots + X_n$ and $\overline{X}_n = S_n/n$.
(b) Is it true that:
(i) $S_n = O_p(1)$? (ii) $S_n = O_p(n^{1/2})$? (iii) $\overline{X}_n = O_p(n^{-1/2})$?
(iv) $n^{1/2}(\overline{X}_n - \mu) = O_p(1)$?
(v) $\sin(S_n) = O_p(1)$?

4. (30 points) Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let $\{A_n\}$ be a sequence of events, $A_n \subset \Omega$, $A_n \in \mathcal{A}$ for $n = 1, 2, \ldots$, and let $X_n = 1_{A_n}$ be the indicator functions of the events $A_n$. Suppose that $\sum_{n=1}^{\infty} P(A_n) < \infty$. Show that $X_n \to_{a.s.} 0$.

Do **either** problems 5 **or** problem 6.

5. (30 points) Suppose that $X, X_1, X_2, \ldots$ are i.i.d. exponential($\lambda$) random variables: hence the distribution function of all the $X_i$'s is $F(x) = 1 - \exp(-\lambda x)$ for $x \geq 0$. Let $M_n = \max\{X_1, \ldots X_n\}$.
(a) Find $P(\lambda^{-1} < X \leq (5/2)\lambda^{-1})$.
(b) Find the inverse distribution function (or quantile function) $F^{-1}$ corresponding to $F$.
(c) Find the distribution function of $M_n$.
(d) Compute the distribution function of $Y_n = M_n - F^{-1}(1 - 1/n)$, show that $Y_n \rightarrow_d Y$ for some random variable $Y$, and find the limiting distribution.
(e) Compute the density function $f_n$ of $Y_n$ and show that $f_n(t) \rightarrow f(t)$ for each $t \in R$ where $f$ is the density of the random variable $Y$.
(f) What can you conclude from (e) and one of the results from chapter 0?

6. (30 points) Suppose that $X, X_1, X_2, \ldots, X_n$ are independent Poisson($\lambda$) random variables:

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots.$$

(a) Use the weak law of large numbers to show that the random vector

$$\underline{\overline{Y}}_n \equiv \frac{1}{n}\sum_{i=1}^{n}(X_i, 1_{[X_i=0]}, 1_{[X_i=1]})'$$

converges in probability to some vector $(a, b, c)' \equiv \underline{y}$ where $(a, b, c)$ depends on $\lambda$. Give $(a, b, c)$ explicitly in terms of $\lambda$.
(b) Use the multivariate CLT to show that

$$\sqrt{n}(\underline{\overline{Y}}_n - \underline{y}) \rightarrow_d \underline{W} \sim N_3(0, \Sigma)$$

for some covariance matrix $\Sigma$; compute $\Sigma$ explicitly in terms of $\lambda$.
(c) The usual estimator of $\lambda$ is $\widehat{\lambda}_n = \overline{X}_n$. A friend suggests the following alternative estimator of $\lambda$:

$$\tilde{\lambda}_n = \frac{\sum_{i=1}^{n} 1_{[X_i=1]}}{\sum_{i=1}^{n} 1_{[X_i=0]}} = \frac{\overline{Y}_{3,n}}{\overline{Y}_{2,n}}.$$

Is $\tilde{\lambda}_n$ a consistent estimator of $\lambda$? If the answer is yes, find the asymptotic variance of this estimator of $\lambda$.