

Statistics 581, Problem Set 7 Solutions

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1. Compute and plot the *score for location*, $-(f'/f)(x)$ when:
- A. $f(x) = \phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$, (normal or Gaussian);
 - B. $f(x) = \exp(-x)/(1 + \exp(-x))^2$, (logistic);
 - C. $f(x) = \frac{1}{2} \exp(-|x|)$, (double exponential);
 - D. $f = t_k$, the t -distribution with k degrees of freedom;
 - E. $f(x) = \exp(-x) \exp(-\exp(-x))$, Gumbel or extreme value.

Solution: A. For $f(x) = (2\pi)^{-1/2} \exp(-x^2/2)$, it follows that $\log f(x) = -x^2/2 + \text{constant}$ so that $(-f'/f)(x) = x$, $-1 - x(f'/f)(x) = x^2 - 1$.

B. For $f(x) = e^{-x}/(1 + e^{-x})^2$, $\log f(x) = -x - 2 \log(1 + e^{-x})$ and

$$-\frac{f'}{f}(x) = \frac{1 - e^{-x}}{1 + e^{-x}},$$

while

$$-1 - x \frac{f'}{f}(x) = x \frac{1 - e^{-x}}{1 + e^{-x}} - 1 \sim |x| - 1 \quad \text{as} \quad |x| \rightarrow \infty.$$

C. For $f(x) = 2^{-1} \exp(-|x|)$,

$$\log f(x) = -|x| + \text{constant},$$

and

$$-\frac{f'}{f}(x) = \begin{cases} -1 & x < 0 \\ \text{undefined} & x = 0 \\ +1 & x > 0 \end{cases},$$

while

$$-1 - x \frac{f'}{f}(x) = |x| - 1, \quad \text{for} \quad x \neq 0.$$

D. For the t_k distribution, $f(x) = \frac{\Gamma(\frac{1}{2}(k+1))}{\Gamma(\frac{1}{2}k)} \frac{1}{\sqrt{\pi k}} (1 + \frac{x^2}{k})^{-(k+1)/2}$,

$$\log f(x) = -\frac{k+1}{2} \log(1 + \frac{x^2}{k}),$$

and

$$-\frac{f'}{f}(x) = \frac{k+1}{k} \frac{x}{1 + \frac{x^2}{k}},$$

while

$$-1 - x \frac{f'}{f}(x) = k \frac{x^2 - 1}{x^2 + k}.$$

E. For $f(x) = \exp(-x) \exp(-\exp(-x))$,

$$\log f(x) = -x - \exp(-x),$$

and

$$-\frac{f'}{f}(x) = 1 - \exp(-x),$$

while

$$-1 - x\frac{f'}{f}(x) = -1 + x(1 - \exp(-x)).$$

See the plots on the last page.

2. Compute $I_f = \int (f'(x)/f(x))^2 f(x) dx$, the information for location, for each of the densities in problem 1.

Solution: A. In this case $I_f = \int x^2 \phi(x) dx = \text{Var}(Z) = 1$ where $Z \sim N(0, 1)$.

B. For the logistic density the information for location is

$$\begin{aligned} I_f &= \int_{-\infty}^{\infty} \left(\frac{1 - e^{-x}}{1 + e^{-x}} \right)^2 dF(x) \\ &= \int_{-\infty}^{\infty} (2F(x) - 1)^2 dF(x) \\ &= \int_0^1 (2u - 1)^2 du = 4\text{Var}(U) \\ &= 4 \frac{1}{12} = \frac{1}{3}. \end{aligned}$$

C. For the double-exponential density, $[(-f'/f)(x)]^2 = 1$, so $I_f = 1$.

D. For the t - distribution with k degrees of freedom, by using a change of variables and letting T_r denote a random variable with the t - distribution with r degrees of freedom,

$$\begin{aligned} I_f &= \int_{-\infty}^{\infty} \left(\frac{k+1}{k} \right)^2 \frac{x^2}{(1+x^2/k)^2} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{k}{2})\sqrt{\pi k}} \frac{1}{(1+x^2/k)^{(k+1)/2}} dx \\ &= \frac{(k+1)(k+2)}{(k+4)(k+3)} \text{Var}(T_{k+4}) \\ &= \frac{(k+1)(k+2)}{(k+4)(k+3)} \frac{k+4}{k+2} = \frac{k+1}{k+3} \end{aligned}$$

since $\text{Var}(T_r) = r/(r-2)$ for $r > 2$.

E. For the extreme value distribution $F(x) = \exp(-\exp(-x))$ and therefore if $X \sim F$, the random variable $Y \equiv \exp(-X) \sim \text{exponential}(1)$:

$$\begin{aligned} P(Y \geq y) &= P(\exp(-X) \geq y) = P(X \leq -\log(y)) \\ &= \exp(-\exp(\log(y))) = \exp(-y). \end{aligned}$$

Since $-(f'/f)(x) = -1 + e^{-x}$, it is easy to see that

$$I_f = E\left[-\frac{f'}{f}(X)\right]^2 = E[\exp(-X) - 1]^2 = E[Y - 1]^2 = \text{Var}(Y) = 1.$$

3. Lehmann and Casella, TPE, Problem 6.5, part (a), page 142.

Solution: 1. If $X \sim N(\xi, \sigma^2)$ where $\theta = (\xi, \sigma)$, then $p_\theta(x) = \sigma^{-1} \phi((x - \xi)/\sigma)$ where $\phi(z) = (2\pi)^{-1/2} \exp(-z^2/2)$ is the standard normal density. Hence

$$\log p_\theta(x) = -\log \sigma + \log \phi\left(\frac{x - \xi}{\sigma}\right),$$

so that

$$\dot{l}_\xi(x) = -\frac{\phi'}{\phi}\left(\frac{x - \xi}{\sigma}\right) \frac{1}{\sigma} = \frac{x - \xi}{\sigma} \cdot \frac{1}{\sigma},$$

and

$$\begin{aligned} \dot{l}_\sigma(x) &= -\frac{1}{\sigma} - \frac{\phi'}{\phi}\left(\frac{x - \xi}{\sigma}\right) \left(\frac{x - \xi}{\sigma}\right) \frac{1}{\sigma} \\ &= \frac{1}{\sigma} \left\{ -1 + \left(\frac{x - \xi}{\sigma}\right)^2 \right\} \end{aligned}$$

where $(X - \xi)/\sigma \sim N(0, 1)$. Hence

$$E_\theta \dot{l}_{\xi^2}(X) = \frac{1}{\sigma^2} E(Z^2) = \frac{1}{\sigma^2},$$

$$E_\theta \dot{l}_\sigma^2(X) = \frac{1}{\sigma^2} E(Z^2 - 1)^2 = \frac{2}{\sigma^2},$$

and

$$E_\theta \dot{l}_\xi(X) \dot{l}_\sigma(X) = \frac{1}{\sigma^2} E\{Z(Z^2 - 1)\} = \frac{1}{\sigma^2} \{E(Z^3) - E(Z)\} = 0.$$

Thus the information matrix is as claimed, and the inverse information matrix is diagonal with diagonal entries σ^2 and $\sigma^2/2$. Note that the information for σ^2 (as opposed to σ) is $1/(2\sigma^4)$, in agreement with our earlier result showing that $\sqrt{n}(S_n^2 - \sigma^2) \rightarrow_d N(0, 2\sigma^4)$ when $X_i \sim N(\mu, \sigma^2)$.

(b) If $X \sim \text{Gamma}(\alpha, \beta)$, with Lehmann's parametrization given on page 27, then

$$p_\theta(x) = \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} \exp(-x/\beta) 1_{(0, \infty)}(x),$$

and

$$\log p_\theta(x) = (\alpha - 1) \log x - \frac{x}{\beta} - \log \Gamma(\alpha) - \alpha \log \beta,$$

and

$$\dot{l}_\alpha(x) = \log x - \log \beta - \psi(\alpha) = \log\left(\frac{x}{\beta}\right) - \psi(\alpha)$$

and

$$\dot{l}_\beta(x) = \frac{x}{\beta^2} - \frac{\alpha}{\beta} = \frac{1}{\beta} \left(\frac{x}{\beta} - \alpha\right)$$

where $X/\beta \sim \text{Gamma}(\alpha, 1)$. Hence

$$\ddot{l}_{\alpha\alpha}(x) = -\psi'(\alpha),$$

$$\ddot{l}_{\beta\beta}(x) = -\frac{2x}{\beta^3} + \frac{\alpha}{\beta^2},$$

$$\ddot{l}_{\alpha\beta} = -\frac{1}{\beta}.$$

Since $E_\theta X = \alpha\beta$, it follows that $-E_\theta \ddot{l}_{\alpha\alpha}(X) = \psi'(\alpha)$, $-E_\theta \ddot{l}_{\beta\beta}(X) = \alpha/\beta^2$, $-E_\theta \ddot{l}_{\alpha\beta}(X) = 1/\beta$, and hence the information matrix is as claimed.

[Note that the determinant of the information matrix is $\beta^{-2}\{\alpha\psi'(\alpha) - 1\}$; this is always positive because of facts about ψ and ψ' : in particular, from Abramowitz and Stegun, 6.4.12, page 260,

$$\alpha\psi'(\alpha) - 1 \sim \frac{1}{2\alpha} + \frac{1}{6\alpha^2} + \dots.$$

Also note that the above computations imply that

$$E_\theta \log X = \log(\beta) + \psi(\alpha), \quad \text{and} \quad \text{Var}_\theta(\log X) = \psi'(\alpha).$$

(c) If $X \sim \text{Beta}(\alpha, \beta)$, then

$$\log p_\theta(x) = \log \Gamma(\alpha + \beta) - \log \Gamma(\alpha) + (\alpha - 1) \log x + (\beta - 1) \log(1 - x),$$

and

$$\begin{aligned} \dot{l}_\alpha(x) &= \log x - (\psi(\alpha) - \psi(\alpha + \beta)), \\ \dot{l}_\beta(x) &= \log(1 - x) - (\psi(\beta) - \psi(\alpha + \beta)). \end{aligned}$$

Hence it follows that

$$\begin{aligned} \ddot{l}_{\alpha\alpha}(x) &= -\psi'(\alpha) + \psi'(\alpha + \beta), \\ \ddot{l}_{\beta\beta}(x) &= -\psi'(\beta) + \psi'(\alpha + \beta), \\ \ddot{l}_{\alpha\beta}(x) &= \psi'(\alpha + \beta), \end{aligned}$$

which are all constant in x . Hence the information matrix is as claimed. Note that the determinant equals

$$\begin{aligned} &(\psi'(\alpha) - \psi'(\alpha + \beta))(\psi'(\beta) - \psi'(\alpha + \beta)) - \{\psi'(\alpha + \beta)\}^2 \\ &= \psi'(\alpha)\psi'(\beta) - (\psi'(\alpha) + \psi'(\beta))\psi'(\alpha + \beta) \\ &> 0 \end{aligned}$$

since ψ is strictly concave (with $\psi' > 0$ and $\psi'' < 0$).

4. Suppose that $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$, $\Theta \subset R^k$ is a parametric model satisfying the hypotheses of the multiparameter Cramér - Rao inequality. Partition θ as $\theta = (\nu, \eta)$ where $\nu \in R^m$ and $\eta \in R^{k-m}$ and $1 \leq m < k$. Let $\dot{l} = \dot{l}_\theta = (\dot{l}_1, \dot{l}_2)$ be the corresponding partition of the (vector of) scores \dot{l} , and, with $\tilde{l} \equiv I^{-1}(\theta)\dot{l}$, the *efficient influence function* for θ , let $\tilde{l} = (\tilde{l}_1, \tilde{l}_2)$ be the corresponding partition of \tilde{l} . In both cases, \dot{l}_1, \tilde{l}_1 are m -vectors of functions, and \dot{l}_2, \tilde{l}_2 are $k - m$ vectors. Partition $I(\theta)$ and $I^{-1}(\theta)$ correspondingly as

$$I(\theta) = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$$

where I_{11} is $m \times m$, I_{12} is $m \times (k - m)$, I_{21} is $(k - m) \times m$, I_{22} is $(k - m) \times (k - m)$. Also write

$$I^{-1}(\theta) = [I^{ij}]_{i,j=1,2}.$$

Verify that:

$$A. I^{11} = I_{11.2}^{-1} \text{ where } I_{11.2} \equiv I_{11} - I_{12}I_{22}^{-1}I_{21},$$

$$I^{22} = I_{22.1}^{-1} \text{ where } I_{22.1} \equiv I_{22} - I_{21}I_{11}^{-1}I_{12},$$

$$I^{12} = -I_{11.2}^{-1}I_{12}I_{22}^{-1},$$

$$I^{21} = -I_{22.1}^{-1}I_{21}I_{11}^{-1}.$$

This amounts to formulas (3) and (4) of section 3.2, page 14.

B. Verify that

$$\tilde{l}_1 = I^{11}\dot{l}_1 + I^{12}\dot{l}_2 = I_{11.2}^{-1}(\dot{l}_1 - I_{12}I_{22}^{-1}\dot{l}_2), \text{ and}$$

$$\tilde{l}_2 = I^{21}\dot{l}_1 + I^{22}\dot{l}_2 = I_{22.1}^{-1}(\dot{l}_2 - I_{21}I_{11}^{-1}\dot{l}_1).$$

Solution: A. This is just block inversion/multiplication of matrices:

$$\begin{aligned} \begin{pmatrix} I^{11} & I^{12} \\ I^{21} & I^{22} \end{pmatrix} \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix} &= \begin{pmatrix} I_{11.2}^{-1} & -I_{11.2}^{-1}I_{12}I_{22}^{-1} \\ -I_{22.1}^{-1}I_{21}I_{11}^{-1} & I_{22.1}^{-1} \end{pmatrix} \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix} \\ &= \begin{pmatrix} I_{11.2}^{-1}(I_{11} - I_{12}I_{22}^{-1}I_{21}) & I_{11.2}^{-1}(I_{12} - I_{12}I_{22}^{-1}I_{22}) \\ I_{22.1}^{-1}(-I_{21} + I_{21}) & I_{22.1}^{-1}(-I_{21}I_{11}^{-1}I_{12} + I_{22}) \end{pmatrix} \\ &= \begin{pmatrix} Ident & 0 \\ 0 & Ident \end{pmatrix} = Identity. \end{aligned}$$

by using the definition of $I_{11.2}$ and $I_{22.1}$.

B. This follows immediately from the formulas for I^{11} and I^{12} by just plugging into the formula $\tilde{l}_1 = I^{11}\dot{l}_1 + I^{12}\dot{l}_2$ for \tilde{l}_1 :

$$\begin{aligned} \tilde{l}_1 &= I_{11.2}^{-1}\dot{l}_1 - I_{11.2}^{-1}I_{12}I_{22}^{-1}\dot{l}_2 \\ &= I_{11.2}^{-1}(\dot{l}_1 - I_{12}I_{22}^{-1}\dot{l}_2) = I_{11.2}^{-1}l_1^*. \end{aligned}$$

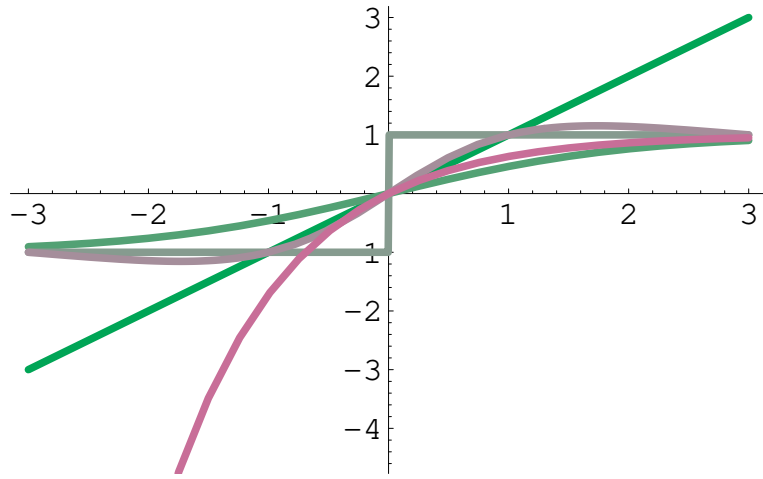


Figure 1: Scores for location.

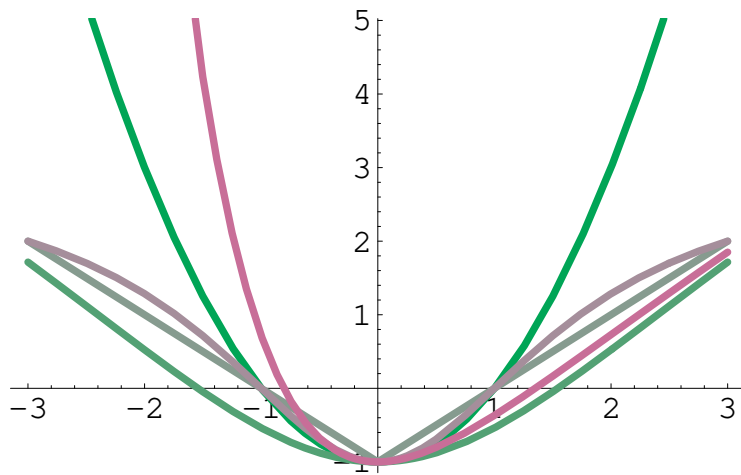


Figure 2: Scores for scale.