Statistics 581
Solutions, Problem Set 5
Wellner; 11/3/00

1. Suppose that $Y_i = \alpha + \theta'(x_i - \overline{x}) + \epsilon_i$, $i = 1, \ldots, n$, where $\epsilon_i \sim (0, \sigma^2)$ are i.i.d. and the x_i's are known vectors in R^k. Equivalently, $\underline{Y} = X\underline{\beta} + \underline{\epsilon}$ where

$$X^T = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 - \overline{x} & x_2 - \overline{x} & \cdots & x_n - \overline{x} \end{pmatrix}$$

so that X is an $n \times (k+1)$ matrix. Let $\underline{\hat{\beta}}$ be the least squares estimator of $\underline{\beta} = (\alpha, \theta')'$; i.e. $\underline{\hat{\beta}} = (X^T X)^{-1} X^T \underline{Y}$. Suppose that $n^{-1}(X^T X) \to D$ where D is positive definite.
(a) What additional condition(s) do you need to impose to prove that

$$\sqrt{n}(\hat{\beta}_n - \beta) \to_d N_{k+1}(0, \text{“}\textit{something}\text{”})\,?$$

(b) Find “something” in part (a).

Solution: (a) Let $\underline{a} \in I\!R^{k+1}$. Now

$$\begin{aligned} \hat{\beta} &= (X^T X)^{-1} X^T Y \\ &= (X^T X)^{-1} X^T (X\beta + \epsilon) \\ &= \beta + (X^T X)^{-1} X^T \epsilon\,, \end{aligned}$$

so

$$\sqrt{n}(\hat{\beta} - \beta) = \sqrt{n}(X^T X)^{-1} X^T \epsilon \equiv B_n \epsilon$$

where $B_n \equiv \sqrt{n}(X^T X)^{-1} X^T$ is a $(k+1) \times n$ matrix. Thus

$$\begin{aligned} a^T(\sqrt{n}(\hat{\beta} - \beta)) &= a^T B_n \epsilon \equiv b_n^T \epsilon \\ &= \sum_{i=1}^n b_{ni}\epsilon_i \equiv \sum_{i=1}^n X_{ni} \end{aligned}$$

where $b_n^T \equiv a^T B_n$ is an $1 \times n$ vector. Now we compute

$$\mu_{ni} = E(X_{ni}) = b_{ni} \cdot 0, \qquad \sigma_{ni}^2 = Var(X_{ni}) = b_{ni}^2 \sigma^2\,,$$

and hence, using the hypothesized convergence of $n^{-1} X^T X \to D$ in the last line,

$$\begin{aligned} \sigma_n^2 &= \sigma^2 \sum_{i=1}^n b_{ni}^2 = \sigma^2 b_n^T b_n \\ &= \sigma^2 a^T B_n B_n^T a = n\sigma^2 a^T (X^T X)^{-1} (X^T X)(X^T X)^{-1} a \\ &= \sigma^2 a^T (n^{-1} X^T X)^{-1} a \to \sigma^2 a^T D^{-1} a \equiv V^2(a) > 0 \end{aligned}$$

since D is nonsingular. To establish asymptotic normality of $a^T(\sqrt{n}(\hat{\beta}-\beta))/\sigma_n$, it remains to verify the Lindeberg condition: namely

$$\frac{1}{\sigma_n}\sum_{i=1}^n E\left\{|X_{ni}|^2 1_{[|X_{ni}|>\delta\sigma_n]}\right\} \to 0 \tag{0.1}$$

for every $\delta > 0$. But, as we have seen before, this holds if

$$\max_{1\le i\le n}|b_{ni}| \to 0 \qquad \text{as} \qquad n\to\infty : \tag{0.2}$$

the left side of (0.1) is bounded as follows:

$$\begin{aligned}
&\frac{1}{\sigma_n^2}\sum_{i=1}^n b_{ni}^2 E\left\{\epsilon_1^2 1_{[|\epsilon_1|>\delta\sigma_n/|b_{ni}|]}\right\} \\
&\le \frac{1}{\sigma^2}E\left\{\epsilon_1^2 1_{[|\epsilon_1|>\delta\sigma_n/\max_{1\le i\le n}|b_{ni}|]}\right\} \\
&\to \frac{1}{\sigma^2}\cdot 0 = 0
\end{aligned}$$

by (0.2), $E(\epsilon_1^2)<\infty$, and the dominated convergence theorem. Thus it follows from the Lindeberg-Feller CLT that

$$a^T(\sqrt{n}(\hat{\beta}-\beta))/\sigma_n \to_d N(0,1),$$

and since $\sigma_n^2 \to \sigma^2 a^T D^{-1}a$, this implies that

$$a^T(\sqrt{n}(\hat{\beta}-\beta)) \to_d N(0, a^T(\sigma^2 D^{-1})a)\,,$$

which in turn, via the Cramér-Wold device, implies

$$\sqrt{n}(\hat{\beta}-\beta) \to_d N_{k+1}(0,\sigma^2 D^{-1})\,.$$

2. Suppose that $X_1,\ldots,X_n$ are i.i.d. Cauchy$(0,1)$; so the density of each X_i with respect to Lebesgue measure on R is $f(x)=\pi^{-1}(1+x^2)^{-1}$, $x\in R$.
(a) Compute the distribution function F of the X_i's.
(b) Compute and plot the inverse distribution function F^{-1} corresponding to F.
(c) For what values of $r>0$ is $E|X_1|^r<\infty$?
(d) Find the distribution function of $M_n \equiv \max_{1\le i\le n} X_i$.
(e) For what values of r is $E|M_n|^r<\infty$?
(f) Find a sequence of constants b_n so that $M_n/b_n \to_d$ and find the limiting distribution. [Hint: see Ferguson, ACLST, Theorem 14, page 95.]

Solution: (a) $F(x)=(1/\pi)\int_{-\infty}^x(1+t^2)^{-1}dt = (1/\pi)\{\arctan(x)+\pi/2\}$.
(b) Setting $F(x)=u$ and solving for $x=F^{-1}(u)$ yields $F^{-1}(u)=\tan(\pi(u-1/2))$. Note that $F^{-1}(1/2)=\tan(0)=0$; $F^{-1}(1)=\tan(\pi/2)=\infty$, and $F^{-1}(0)=\tan(-\pi/2)=-\infty$.
(c) We compute

$$E|X_1| = \frac{1}{\pi}\int_{-\infty}^{\infty}|x|^r\frac{1}{1+x^2}dx$$

$$
\begin{aligned}
&= \frac{2}{\pi}\left\{\int_0^1 \frac{x^r}{1+x^2}dx + \int_1^\infty \frac{x^r}{1+x^2}dx\right\} \\
&\le \frac{2}{\pi}\left\{\int_0^1 \frac{x^r}{1+x^2}dx + \int_1^\infty \frac{x^r}{x^2}dx\right\} \\
&= \frac{2}{\pi}\left\{\int_0^1 \frac{x^r}{1+x^2}dx + \frac{1}{1-r}\right\} < \infty
\end{aligned}
$$

if $r < 1$. Since

$$
E|X_1| = \frac{2}{\pi}\int_0^\infty \frac{x}{1+x^2}dx = \infty\,,
$$

$E|X_1|^r < \infty$ if and only if $r < 1$.
(d) Since the X_i's are i.i.d. with distribution function F,

$$
F_{M_n}(x) = P(M_n \le x) = P(X_1 \le x, \ldots, X_n \le x) = F(x)^n.
$$

(e) First, note that

$$
1 - F_{|M_n|}(x) = P(|M_n| > x) = P(\cup_{i=1}^n [|X_i| > x]) \le \sum_{i=1}^n P(|X_i| > x) = n(1 - F_{|X_1|}(x))
$$

where $F_{|X_1|}(x) = P(|X_1| \le x) = F(x) - F(-x)$. Hence

$$
\begin{aligned}
E|M_n|^r &= \int_0^\infty r t^{r-1}(1 - F_{|M_n|}(t))dt \\
&\le \int_0^\infty r t^{r-1} n (1 - F_{|X|}(t))dt \\
&= nE|X_1|^r < \infty
\end{aligned}
$$

if $r < 1$ by part (d). But since $E|M_n|^r \ge E|X_1|^r = \infty$ if $r \ge 1$, we conclude that $E|M_n|^r < \infty$ if and only if $r < 1$.
(f) Note that $1 - F(x) = \pi^{-1}\int_x^\infty (1+t^2)^{-1}dt \sim 1/(\pi x)$ in the sense that $x(1-F(x)) \to 1/\pi$ as $x \to \infty$. [This follows easily by writing the left side as $(1-F(x))/(x^{-1})$ and using L'Hopital's rule.] Hence for $b_n \to \infty$ and $x > 0$

$$
F_{M_n/b_n}(x) = P(M_n \le x b_n) = F(x b_n)^n \qquad \text{by part d}
$$

and, with $c_n \equiv x b_n(1 - F(x b_n)) \to 1/\pi$,

$$
\begin{aligned}
F_{M_n/b_n}(x) &= F(xb_n)^n = (1 - (1 - F(xb_n)))^n \\
&= (1 - [xb_n(1 - F(xb_n))]/(xb_n))^n \\
&= (1 - c_n/xb_n)^n.
\end{aligned}
$$

From this last expression it becomes clear that the choice $b_n = n$ yields,

$$
F_{M_n/b_n}(x) \to \exp(-1/\pi x) \equiv G(x), \qquad \text{for } x > 0,
$$

while for $x \le 0$

$$
F_{M_n/b_n}(x) \to 0
$$

since $F(xb_n) \le 1/2$ for $x \le 0$. Note that $G(0) = \exp(-\infty) = 0$, G is monotone increasing, and $G(\infty) = \exp(0) = 1$. In fact, G is a member of the Weibull family with shape parameter -1, and is one of the three different families that can arise as limit distributions of maxima of independent rv's; see e.g. Ferguson (1996), *A Course in Large Sample Theory*, page 95.

3. Suppose that $X_1, \ldots, X_n$ are i.i.d. with continuous distribution function F. Let F_0 be a fixed, specified distribution function. Suppose we want to test $H: \ F = F_0$ versus $K: \ F \ne F_0$. Consider the *Cramér - von Mises statistic* given by

$$C_n^2 \equiv \int_{-\infty}^{\infty} n(\mathbb{F}_n(x) - F_0(x))^2 dF_0(x) .$$

(a) Show that when the null hypothesis is true, $F = F_0$, then

$$C_n^2 =_d \int_0^1 n(\mathbb{G}_n(t) - t)^2 dt ,$$

where $\mathbb{G}_n$ is the empirical d.f. of n i.i.d. Uniform$(0,1)$ rv's.
(b) Show that when the null hypothesis is true,

$$C_n^2 \to_d \int_0^1 \mathbb{U}(t)^2 dt$$

where $\mathbb{U}$ is a standard Brownian bridge process.
[Hint: Use the fact that $\mathbb{U}_n \Rightarrow \mathbb{U}$ in $(D[0,1], \|\cdot\|_\infty)$ and the continuous mapping theorem.]
(c) Suppose that the null hypothesis fails. Thus $F \ne F_0$. Show that in this case

$$n^{-1} C_n^2 \to_{a.s.} \int_{-\infty}^{\infty} (F(x) - F_0(x))^2 dF_0(x) > 0 ,$$

and hence the test based on C_n^2 is consistent for all $F \ne F_0$.
(d) Suppose that $F = F_n$ satisfies $\sqrt{n}(F_n(x) - F_0(x)) \to g(x)$ in $L_2(F_0)$; i.e.

$$\int [\sqrt{n}(F_n(x) - F_0(x)) - g(x)]^2 dF_0(x) \to 0 .$$

Describe the limiting distribution of C_n^2 under the local alternatives F_n in terms of a Brownian bridge process $\mathbb{U}$ and g.

Soluton: (a) Since $\mathbb{F}_n(x) =_d \mathbb{F}_n^*(x) = \mathbb{G}_n(F_0(x))$ when $F = F_0$ and F_0 is continuous, it follows that

$$\begin{aligned} C_n^2 &=_d \int n(\mathbb{F}_n^*(x) - F_0(x))^2 dF_0(x) \\ &= \int n(\mathbb{G}_n(F_0(x)) - F_0(x))^2 dF_0(x) \\ &= \int_0^1 n(\mathbb{G}_n(t) - t)^2 \, dt \end{aligned}$$

by using the change of variables $t = F_0(x)$ in the last line.
(b) Note that if $\{x_n\}$ is a sequence of functions in $D[0,1]$ satisfying $\|x_n - x\|_\infty \equiv \sup_{0 \le t \le 1} |x_n(t) - x(t)| \to 0$, then with $g(x) \equiv \int_0^1 x^2(t)dt$, $g(x_n) \to g(x)$. It follows from (a) and the continuous mapping theorem that, under the null hypothesis,

$$C_n^2 = \int_0^1 [\mathbb{U}_n(t)]^2 \, dt \to_d \int_0^1 [\mathbb{U}(t)]^2 \, dt \equiv C^2 \, .$$

The distribution of C^2 is the same as that of $\sum_{j=1}^\infty Z_j^2/(\pi^2 j^2)$ where Z_j are i.i.d. $N(0,1)$, and tables of the d.f. are available; see e.g. Shorack and Wellner (1986), page 147.
(c) When the null hypothesis fails (so $F \ne F_0$),

$$\begin{aligned} n^{-1}C_n^2 &= \int [\mathbb{F}_n(x) - F_0(x)]^2 dF_0(x) \\ &=_d \int [\mathbb{G}_n(F(x)) - F_0(x)]^2 dF_0(x) \\ &\to_{a.s.} \int [F(x) - F_0(x)]^2 dF_0(x) \equiv c^2 \end{aligned}$$

since $\|\mathbb{G}_n - I\|_\infty \to_{a.s.} 0$.
(d) If $F = F_n$ satisfies $\sqrt{n}(F_n - F_0) \to g$, then

$$\begin{aligned} C_n^2 &=_d \int [\sqrt{(}\mathbb{F}_n^*(x) - F_n(x)) + \sqrt{n}(F_n(x) - F_0(x))]^2 dF_0(x) \\ &= \int [\mathbb{U}_n(F_n(x)) + \sqrt{n}(F_n - F_0)]^2 dF_0(x) \\ &\to_d \int [\mathbb{U}(F_0(x)) + g(x)]^2 dF_0(x) \\ &= \int_0^1 [\mathbb{U}(t) + g(F_0^{-1}(t))]^2 \, dt \, . \end{aligned}$$

4. Suppose that $X_1, \ldots, X_n$ are i.i.d. with the Weibull distribution F_θ given by

$$1 - F_\theta(x) = \exp(-(x/\alpha)^\beta), \qquad x \ge 0$$

where $\theta = (\alpha, \beta) \in (0,\infty) \times (0,\infty)$.
(a) Find the inverse (or quantile function) $F_\theta^{-1}(u)$ corresponding to F_θ in terms of α, β, and $u \in (0,1)$, and show that

$$\log F_\theta^{-1}(u) = \log \alpha + \frac{1}{\beta} \log \log \left(\frac{1}{1-u} \right) .$$

(b) Fix $t \in (0, 1/2)$. Use the t−th and $(1-t)$−th quantiles of the X_i's, namely $\mathbb{F}_n^{-1}(t)$ and $\mathbb{F}_n^{-1}(1-t)$, to obtain simple consistent estimators $\hat{\alpha}_n$ and $\hat{\beta}_n$ of α and β. Prove that your estimators are consistent.
(c) Prove that your estimators $\hat{\alpha}_n$ and $\hat{\beta}_n$ satisfy

$$\sqrt{n} \begin{pmatrix} \hat{\alpha}_n - \alpha \\ \hat{\beta}_n - \beta \end{pmatrix} \to_d N_2(0, \Sigma)$$

and identify Σ as a function of α, β, and t.
(d) How would you choose t to minimize the asymptotic variance of $\hat{\beta}_n$?

Solution: (a) Since $1 - F_\theta(x) = \exp(-(x/\alpha)^\beta)$, it follows we can solve $F_\theta(x) = u$ for $x = F_\theta^{-1}(u)$. This yields

$$F_\theta^{-1}(u) = \alpha(-\log(1-u))^{1/\beta},$$

or

$$\log F_\theta^{-1}(u) = \log\alpha + \frac{1}{\beta}\log\log\left(\frac{1}{1-u}\right). \tag{0.3}$$

(b) Since we can estimate $F_\theta^{-1}(t)$ and $F_\theta^{-1}(1-t)$ respectively by $\mathbb{F}_n^{-1}(t)$ and $\mathbb{F}_n^{-1}(1-t)$ respectively, the relationship in (0.3) suggests that we estimate α and β as the solutions $\hat{\alpha}$ and $\hat{\beta}$ of the pair of equations

$$\log \mathbb{F}_n^{-1}(t) = \log\hat{\alpha} + \frac{1}{\hat{\beta}}\log\log 1/(1-t), \tag{0.4}$$

$$\log \mathbb{F}_n^{-1}(1-t) = \log\hat{\alpha} + \frac{1}{\hat{\beta}}\log\log 1/t. \tag{0.5}$$

Letting $A_t \equiv \log\log 1/(1-t)$, and $B_t \equiv \log\log 1/t$, we find that

$$\begin{aligned} 1/\hat{\beta} &= \frac{1}{B_t - A_t}(\log\mathbb{F}_n^{-1}(1-t) - \log\mathbb{F}_n^{-1}(t)) \\ &\equiv a_t \log\mathbb{F}_n^{-1}(1-t) + b_t\log\mathbb{F}_n^{-1}(t) \end{aligned}$$

and

$$\begin{aligned} \log\hat{\alpha} &= \frac{-A_t}{B_t - A_t}\log\mathbb{F}_n^{-1}(1-t) + \frac{B_t}{B_t - A_t}\log\mathbb{F}_n^{-1}(t)) \\ &\equiv c_t\log\mathbb{F}_n^{-1}(t) + d_t\log\mathbb{F}_n^{-1}(1-t) \end{aligned}$$

where

$$a_t \equiv \frac{1}{B_t - A_t}, \qquad b_t = -a_t, \qquad c_t \equiv -A_t a_t \qquad d_t \equiv B_t a_t.$$

Since $(\mathbb{F}_n^{-1}(t), \mathbb{F}_n^{-1}(1-t)) \to_{a.s.} (F_\theta^{-1}(t), F_\theta^{-1}(1-t))$, It follows easily by the continuous mapping theorem that

$$\frac{1}{\hat{\beta}} \to_{a.s.} a_t\log F_\theta^{-1}(1-t) + b_t\log F_\theta^{-1}(t) = \frac{1}{\beta},$$

and

$$\log\hat{\alpha} \to_{a.s.} c_t\log F_\theta^{-1}(1-t) + d_t\log F_\theta^{-1}(t) = \log\alpha,$$

and hence by the continuous mapping theorem, $(\hat{\alpha}, \hat{\beta}) \to_{a.s.} (\alpha, \beta)$.
(c) First, we know that

$$\sqrt{n}\begin{pmatrix} \mathbb{F}_n^{-1}(1-t) - F^{-1}(1-t) \\ \mathbb{F}_n^{-1}(t) - F^{-1}(t) \end{pmatrix} \to_d \underline{Z} \sim N_2(0,\Sigma)$$

where

$$\Sigma = \begin{pmatrix} \frac{t(1-t)}{f^2(F^{-1}(1-t))} & \frac{t^2}{f(F^{-1}(t))f(F^{-1}(1-t))} \\ \frac{t^2}{f(F^{-1}(t))f(F^{-1}(1-t))} & \frac{t(1-t)}{f^2(F^{-1}(t))} \end{pmatrix}.$$

This implies that

$$\sqrt{n}\begin{pmatrix} \log \mathbb{F}_n^{-1}(1-t) - \log F^{-1}(1-t) \\ \log \mathbb{F}_n^{-1}(t) - \log F^{-1}(t) \end{pmatrix} \to_d D\underline{Z} \sim N_2(0, D\Sigma D^T)$$

where

$$D = \begin{pmatrix} 1/F^{-1}(1-t) & 0 \\ 0 & 1/F^{-1}(t) \end{pmatrix}.$$

Hence it follows that

$$\begin{aligned} & \sqrt{n}\begin{pmatrix} 1/\hat{\beta} - 1/\beta \\ \log\hat{\alpha} - \log\alpha \end{pmatrix} \\ &= M\sqrt{n}\begin{pmatrix} \log \mathbb{F}_n^{-1}(1-t) - \log F^{-1}(1-t) \\ \log \mathbb{F}_n^{-1}(t) - \log F^{-1}(t) \end{pmatrix} \\ &\to_d M D\underline{Z} \sim N_2(0, MD\Sigma D^T M^T). \end{aligned}$$

where

$$M = \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix} = a_t \begin{pmatrix} 1 & -1 \\ -A_t & B_t \end{pmatrix}.$$

Finally, with $g(x,y) = (g_1(x), g_2(y))$, $g_1(x) = 1/x$, $g_2(y) = \exp y$, we find, by the delta-method, that

$$\begin{aligned} & \sqrt{n}\begin{pmatrix} \hat{\beta} - \beta \\ \hat{\alpha} - \alpha \end{pmatrix} \\ &\to_d \nabla g M D\underline{Z} \sim N_2(0, \nabla g M D\Sigma D^T M^T \nabla g^T) \end{aligned}$$

where

$$\nabla g = \begin{pmatrix} \beta^2 & 0 \\ 0 & \alpha \end{pmatrix}.$$

We begin combining all this by noting that $D\Sigma D^T$ involves the function

$$\begin{aligned} F^{-1}(u)f(F^{-1}(u)) &= \alpha\left(\log\left(\frac{1}{1-u}\right)\right)^{1/\beta} \frac{\beta}{\alpha}\left(\log\left(\frac{1}{1-u}\right)\right)^{(\beta-1)/\beta}(1-u) \\ &= \beta(1-u)\log\left(\frac{1}{1-u}\right) \end{aligned}$$

at the points $u = t$ and $u = 1-t$. Computing $D\Sigma D^T$ yields

$$\begin{aligned} D\Sigma D^T &= \beta^{-2}\begin{pmatrix} \frac{1-t}{t(\log(1/t))^2} & \frac{t}{(1-t)\log(1/t)\log(1/(1-t))} \\ \frac{t}{(1-t)\log(1/t)\log(1/(1-t))} & \frac{t}{(1-t)(\log(1/(1-t)))^2} \end{pmatrix} \\ &\equiv \beta^{-2}\begin{pmatrix} s_{11}(t) & s_{12}(t) \\ s_{12}(t) & s_{22}(t) \end{pmatrix}. \end{aligned}$$

Since the matrix M just depends on t, we find that the matrix

$$MD\Sigma D^T M^T = \beta^{-2} a_t^2 \begin{pmatrix} r_{11}(t) & r_{12}(t) \\ r_{12}(t) & r_{22}(t) \end{pmatrix},$$

where

$$\begin{aligned} r_{11}(t) &= s_{11}(t) - 2s_{12}(t) + s_{22}(t) \\ r_{12}(t) &= B_t(s_{12}(t) - s_{22}(t)) - A_t(s_{11}(t) - s_{12}(t)) \\ r_{22}(t) &= A_t^2 s_{11}(t) - 2A_t B_t s_{12}(t) + B_t^2 s_{22}(t)\,. \end{aligned}$$

Thus we conclude that the asymptotic covariance matrix of $(\hat{\beta}, \hat{\alpha})$ is given by

$$\nabla g M D\Sigma D^T M^T \nabla g^T = a_t^2 \begin{pmatrix} \beta^2 r_{11}(t) & \alpha r_{12}(t) \\ \alpha r_{12}(t) & (\alpha/\beta)^2 r_{22}(t) \end{pmatrix}.$$

(d) The asymptotic variance of $\hat{\beta}$ is

$$\beta^2 a_t^2 r_{11}(t) = \beta^2 \left(s_{11}(t) - 2s_{12}(t) + s_{22}(t)\right) a_t^2\,.$$

This is minimized by $t = t_0 \approx .10725$, and the minimum value is $\beta^2(1.13264) > \beta^2(6/\pi^2)$ see Figures 1 and 2 below. This ad-hoc estimator $\hat{\beta}$ based on quantiles is *inefficient*; its asymptotic variance (for any value of t, includinjg the minimizing t_0) is larger than the best possible asymptotic variance, which is $\beta^2(6/\pi^2)$ as we will see in Chapter 3.)

The asymptotic variance of $\hat{\alpha}$ is

$$(\alpha/\beta)^2 a_t^2 r_{22}(t) = (\alpha/\beta)^2 \left(A_t^2 s_{11}(t) - 2A_t B_t s_{12}(t) + B_t^2 s_{22}(t)\right).$$

This is minimized by $t = t_0 \approx .2295$, and the minimum value is $(\alpha/\beta)^2(1.423) > (\alpha/\beta)^2(1.11)$ see Figures 3 and 4 below. This ad-hoc estimator $\hat{\beta}$ based on quantiles is also *inefficient*; its asymptotic variance (for any value of t, including the minimizing t_0) is larger than the best possible asymptotic variance, which is about $(\alpha/\beta)^2(1.11)$ as we will see in Chapter 3.

Plots for Problem 4, part (d)

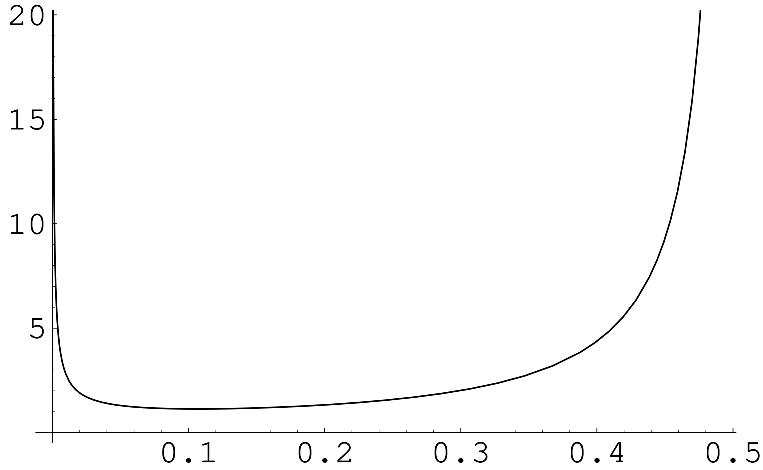

Figure 1: Variance of $\hat{\beta}$, $0 \leq t < .5$.

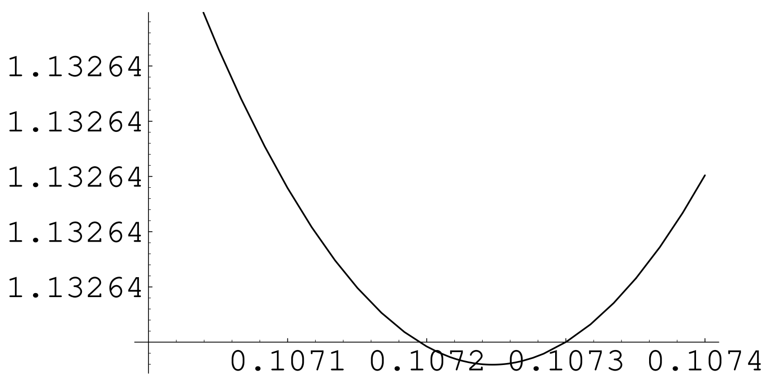

Figure 2: Variance of $\hat{\beta}$, $.1070 \leq t < .1074$.

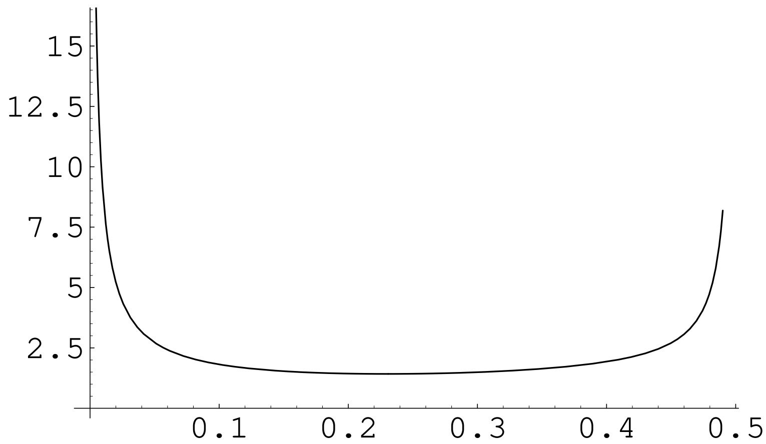

Figure 3: Variance of $\hat{\alpha}$, $0 \leq t < .5$.

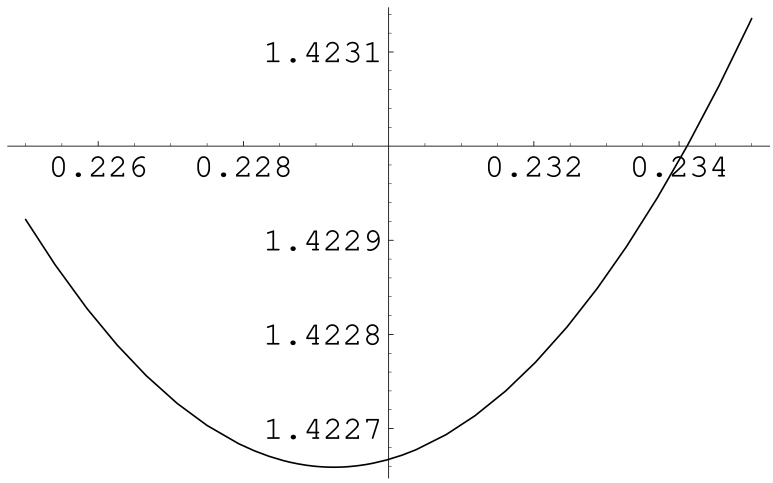

Figure 4: Variance of $\hat{\alpha}$, $.26 \leq t < .32$.