

Statistics 581, Solutions, Problem Set 4

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1. Suppose that $\underline{N}_n \sim \text{Mult}_k(n, \underline{p}_n)$ where $\underline{p}_n = \underline{p}_0 + n^{-1/2}\underline{c}$. Thus $\underline{N}_n = \sum_{i=1}^n \underline{M}_{ni}$ where $\underline{M}_{n1}, \dots, \underline{M}_{nn}$ are i.i.d. $\text{Mult}_k(1, \underline{p}_n)$. Show that with $\widehat{\underline{p}}_n \equiv \underline{N}_n/n$ we have

$$\sqrt{n}(\widehat{\underline{p}}_n - \underline{p}_n) \rightarrow_d N_k(0, \Sigma)$$

where $\Sigma = \text{diag}(\underline{p}_0) - \underline{p}_0 \underline{p}_0^T$.

Hint: Use the Cramér-Wold device and the Liapunov CLT.

Solution: Fix $\underline{a} \in R^k$. Then we want to show that

$$\underline{a}^T \sqrt{n}(\widehat{\underline{p}}_n - \underline{p}_n) \rightarrow_d N(0, \underline{a}^T (\text{diag}(\underline{p}_0) - \underline{p}_0 \underline{p}_0^T) \underline{a}).$$

But since $\underline{N}_n = \sum_{i=1}^n \underline{V}_{ni}$ where $\underline{V}_{ni} \sim \text{Mult}_k(1, \underline{p}_n)$ are i.i.d. for each n , we can write

$$\begin{aligned} \underline{a}^T \sqrt{n}(\widehat{\underline{p}}_n - \underline{p}_n) &= \sum_{i=1}^n \sum_{j=1}^k a_j (V_{ni,j} - p_{nj}) / \sqrt{n} \\ &\equiv \sum_{i=1}^n X_{ni} \end{aligned}$$

where the X_{ni} 's have $\mu_{ni} = E(X_{ni}) = 0$,

$$\sigma_{ni}^2 = \text{Var}(X_{ni}) = \underline{a}^T (\text{diag}(\underline{p}_n) - \underline{p}_n \underline{p}_n^T) \underline{a} / n$$

and

$$\gamma_{ni} = E|X_{ni}|^3 = n^{-3/2} \sum_{j'=1}^k \left\{ \left| a_{j'}(1 - p_{nj'}) + \sum_{j=1, j \neq j'}^k a_j(0 - p_{nj}) \right|^3 \right\} p_{nj'}$$

so that

$$\sigma_n^2 = \sum_1^n \sigma_{ni}^2 = \underline{a}^T (\text{diag}(\underline{p}_n) - \underline{p}_n \underline{p}_n^T) \underline{a} \rightarrow \underline{a}^T \Sigma \underline{a}$$

while

$$\begin{aligned} \gamma_n &= \sum_1^n \gamma_{ni} \\ &= n^{-1/2} \sum_{j'=1}^k \left\{ \left| a_{j'}(1 - p_{nj'}) + \sum_{j=1, j \neq j'}^k a_j(0 - p_{nj}) \right|^3 \right\} p_{nj'} \\ &\rightarrow 0 \cdot M(\underline{a}, \underline{p}_0) = 0 \end{aligned}$$

where

$$M(\underline{a}, \underline{p}_0) = \sum_{j'=1}^k \left\{ \left| a_{j'}(1 - p_{0j'}) + \sum_{j=1, j \neq j'}^k a_j(0 - p_{0j}) \right|^3 \right\} p_{0j'}.$$

Hence it follows that $\gamma_n/\sigma_n^{3/2} \rightarrow 0$, and

$$\frac{\underline{a}^T \sqrt{n}(\hat{\underline{p}}_n - \underline{p}_n)}{\sigma_n} = \frac{\sum_{i=1}^n X_{ni}}{\sigma_n} \rightarrow_d N(0, 1).$$

This implies

$$\underline{a}^T \sqrt{n}(\hat{\underline{p}}_n - \underline{p}_n) \rightarrow_d N(0, \underline{a}^T \Sigma \underline{a}),$$

and by Cramér - Wold, this implies

$$\sqrt{n}(\hat{\underline{p}}_n - \underline{p}_n) \rightarrow_d N_k(0, \Sigma).$$

2. Suppose that $\underline{N}_n \sim \text{Mult}_k(n, \underline{p})$, and let $\hat{\underline{p}}_n = \underline{N}_n/n$ as before. Consider the square of the Hellinger distance,

$$H^2(\hat{\underline{p}}_n, \underline{p}_0) = \sum_{j=1}^k (\sqrt{\hat{p}_j} - \sqrt{p_{0j}})^2.$$

- (a) Show that if the null hypothesis $\underline{p} = \underline{p}_0$ holds, then $4nH^2(\hat{\underline{p}}_n, \underline{p}_0) \rightarrow_d \chi_{k-1}^2$.
 (b) Show that if $\underline{p} \neq \underline{p}_0$ holds, then $4H^2(\hat{\underline{p}}_n, \underline{p}_0) \rightarrow_p$ “something” > 0 , and find “something”.
 (c) Show that if $\underline{p}_n = \underline{p}_0 + n^{-1/2} \underline{c}$ holds, with $\underline{1}' \underline{c} = 0$, then $4nH^2(\hat{\underline{p}}_n, \underline{p}_0) \rightarrow_d \chi_{k-1}^2(\delta)$ where $\delta = \sum_{i=1}^k c_i^2/p_{i0}$.

Solution: (a) Let $\underline{Z}_n \equiv \sqrt{n}(\hat{\underline{p}}_n - \underline{p}_0)$. Then $\underline{Z}_n \rightarrow_d \underline{Z} \sim N_k(0, \Sigma)$ with $\Sigma = \text{diag}(\underline{p}_0) - \underline{p}_0 \underline{p}_0^T$. Thus, by the delta - method,

$$\begin{aligned} \underline{Y}_n &\equiv 2\sqrt{n}(\sqrt{\hat{\underline{p}}_n} - \sqrt{\underline{p}_0}) \\ &\rightarrow_d \text{diag}(1/\sqrt{\underline{p}_0}) \underline{Z} \equiv \underline{Y} \sim N_k(0, I - \sqrt{\underline{p}_0} \sqrt{\underline{p}_0^T}) \end{aligned}$$

Hence, by the continuous mapping theorem,

$$4nH^2(\hat{\underline{p}}_n, \underline{p}_0) = \underline{Y}_n^T \underline{Y}_n \rightarrow_d \underline{Y}^T \underline{Y}.$$

It remains to answer the question: what is the distribution of $\underline{Y}^T \underline{Y}$? This goes just exactly as in the case of the limit for the chi-square statistic Q_n . Let Γ be an orthogonal matrix with first row $\sqrt{\underline{p}_0^T}$. Then

$$\Gamma \underline{Y} \sim N_k(0, \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}),$$

which has first coordinate 0, and the remaining $k - 1$ coordinates are i.i.d. $N(0, 1)$. Further, $\Gamma^T \Gamma = I$ and hence

$$\underline{Y}^T \underline{Y} = \underline{Y}^T \Gamma^T \Gamma \underline{Y} = (\Gamma \underline{Y})^T (\Gamma \underline{Y}) \sim \chi_{k-1}^2.$$

Thus with $H_n^2 \equiv H^2(\hat{\underline{p}}_n, \underline{p}_0)$,

$$4nH_n^2 \rightarrow_d \underline{Y}^T \underline{Y} \sim \chi_{k-1}^2.$$

(b) Under fixed $\underline{p} \neq \underline{p}_0$, $\hat{p}_n \rightarrow_{a.s.} p$. Hence by the continuous mapping theorem

$$\begin{aligned} 4H_n^2 &= 4 \sum_{j=1}^k \left\{ \sqrt{\hat{p}_j} - \sqrt{p_{j0}} \right\}^2 \\ &\rightarrow_{a.s.} 4 \sum_{j=1}^k (\sqrt{p_j} - \sqrt{p_{j0}})^2 \\ &= 4H^2(p, p_0) > 0. \end{aligned}$$

Therefore, under $p \neq p_0$, $H_n^2 \rightarrow_{a.s.} \infty$, and hence

$$P_p(4nH_n^2 \geq \chi_{k-1, \alpha}^2) \rightarrow 1.$$

(c) Under local alternatives, Liapunov's CLT, the Cramér - Wold device, and the delta method, yield

$$\begin{aligned} \underline{Y}_n &= 2\sqrt{n}(\sqrt{\hat{\underline{p}}_n} - \sqrt{\underline{p}_n}) + 2\sqrt{n}(\sqrt{\underline{p}_n} - \sqrt{\underline{p}_0}) \\ &\rightarrow_d \underline{Y} + \text{diag}(1/\sqrt{\underline{p}_0})\underline{c} \\ &\equiv \underline{Y} + \underline{\mu} \\ &\sim N_k(\underline{\mu}, I - \sqrt{p_0}\sqrt{p_0}^T). \end{aligned}$$

Now with Γ as in part (a)

$$\Gamma(\underline{Y} + \underline{\mu}) = \Gamma \underline{Y} + \Gamma \underline{\mu} = \Gamma \underline{Y} + \underline{b}$$

where the first coordinate of \underline{b} is 0. Thus $\Gamma \underline{Y} + \underline{b}$ has first coordinate 0, and the remaining $k - 1$ coordinates are independent $N(b_i, 1)$. Hence

$$\begin{aligned} (\underline{Y} + \underline{\mu})^T (\underline{Y} + \underline{\mu}) &= (\Gamma \underline{Y} + \underline{b})^T (\Gamma \underline{Y} + \underline{b}) \\ &\sim \chi_{k-1}^2(\underline{b}^T \underline{b}) = \chi_{k-1}^2\left(\sum_{j=1}^k c_j^2 / p_{j0}\right) \end{aligned}$$

Thus the local asymptotic power of the test based on the Hellinger statistics $4nH_n^2$ is the same as that of the chi-square statistic Q_n .

3. Lehmann & Casella (1998), problem 5.22, page 406, define a noncentral chi-square distribution as follows: $Y \sim \chi_r^2(\lambda)$ if $(Y|K = k) \sim \chi_{2k+r}^2$ and $K \sim \text{Poisson}(\lambda)$. If $X \sim N_r(\underline{\mu}, I)$, what is the distribution of $X^T X = |X|^2$:

- (a) In the notation I have used in class?
 (b) In Lehmann's notation?
 (c) Is there any contradiction or false statement in Lehmann & Casella's problem 5.22? (Note the Poisson($\lambda/2$) in problem 5.23!).

Solution: (a) In the notation I have used in class, the distribution of $|X|^2$ is $\chi_r^2(\delta)$ with $\delta = \mu'\mu = |\mu|^2$. The parameter of the Poisson distribution involved in the conditional description of $\chi_r^2(\delta)$ is $\delta/2 = |\mu|^2/2$.

(b) In the notation of Lehmann & Casella, problem 5.23, page 406, the distribution of $|X|^2$ is $\chi_r^2(|\mu|^2/2)$. Then the parameter of the Poisson distribution is again $|\mu|^2/2$.

(c) The claimed identities do not hold with $\lambda = |\theta|^2$. See the paper by Bock (1975) referenced on page 407 [*Ann. Statist.* **3**, 209-218], and, in particular, Theorem A, page 216, with $h(y) = 1/y$, in which the parameter of the Poisson distribution is clearly $\lambda = |\theta|^2/2$.

4. Suppose that X_1, X_2, \dots are i.i.d. (μ, σ^2) with $\mu_4 < \infty$. Let $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$ and $S_n^2 = (n-1)^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$ be the sample mean and sample variance respectively.

- (a) Show that

$$\sqrt{n} \begin{pmatrix} \bar{X}_n - \mu \\ S_n^2 - \sigma^2 \end{pmatrix} \rightarrow_d \underline{Z} \sim N_2(0, \Sigma)$$

where

$$\begin{pmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{pmatrix}.$$

(b) Suppose $\mu \neq 0$. Use (a) to find the limiting distribution of the sample *coefficient of variation* $C_n \equiv S_n/\bar{X}_n$; i.e. show that $\sqrt{n}(C_n - c) \rightarrow_d N(0, V^2)$ with $c \equiv \sigma/\mu$ and find V^2 .

Solution: (a) Since $S_n^2 = n^{-1} \sum_{i=1}^n (X_i - \mu)^2 + o_p(1/\sqrt{n})$, we have

$$\begin{aligned} \sqrt{n} \begin{pmatrix} \bar{X}_n - \mu \\ S_n^2 - \sigma^2 \end{pmatrix} &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \begin{pmatrix} X_i - \mu \\ (X_i - \mu)^2 - \sigma^2 \end{pmatrix} + o_p(1) \\ &\rightarrow_d \underline{Z} \sim N_2(0, \Sigma) \end{aligned}$$

by the multivariate CLT where Σ is as given above.

(b) The function $g(u, v) = \sqrt{v}/u$ is differentiable at points (u, v) with $u \neq 0$, and the derivative is $\nabla g(u, v) = (-u^{-2}\sqrt{v}, (1/2)v^{-1/2}u^{-1})$ so that $\nabla g(\mu, \sigma^2) = (-\mu^{-2}\sigma, (1/2)\sigma^{-1}\mu^{-1}) = (\sigma/\mu)(-1/\mu, (1/2)\sigma^2)$. Hence it follows from the delta method (g' theorem) that

$$\begin{aligned} \sqrt{n}(C_n - c) &= \sqrt{n}(g(\bar{X}_n, S_n^2) - g(\mu, \sigma^2)) \\ &\rightarrow_d \nabla g \cdot \underline{Z} \sim N(0, \nabla g^T \Sigma \nabla g) \end{aligned}$$

and it is easy to calculate that

$$\begin{aligned} \nabla g^T \Sigma \nabla g &= c^2 \left\{ \frac{\sigma^2}{\mu^2} - \frac{\mu_3}{2\mu\sigma^2} + \frac{1}{2} \left(1 + \frac{\gamma_2}{2} \right) \right\} \\ &= c^2 \left(c^2 - c\gamma_1 + \frac{1}{2} \left(1 + \frac{\gamma_2}{2} \right) \right) \end{aligned}$$

where $\gamma_1 \equiv \mu_3/\sigma^3$. Note that when the X_i 's are normal (so $\gamma_1 = \gamma_2 = 0$), this reduces to $c^2(c^2 + 1/2)$.