Statistics 581, Solutions, Problem Set 1

Wellner; 10/6/00

1. Let X and Y be i.i.d. Uniform$(0,1)$ random variables Define $U = X+Y$, $V = \min(X,Y) = X \wedge Y$.

   (i) What is the range of (U,V)?

   (ii) Find the joint density function $f_{U,V}(u,v)$ of the pair (U,V). Are U and V independent?

Solution: (i) The range of (X,Y) is
$A = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$. The range of (U,V) is

$$B = \{(u,v) : 0 \le u \le 1, 0 \le v \le u/2\} \cup \{(u,v) : 1 < u \le 2, u-1 \le v \le u/2\}.$$

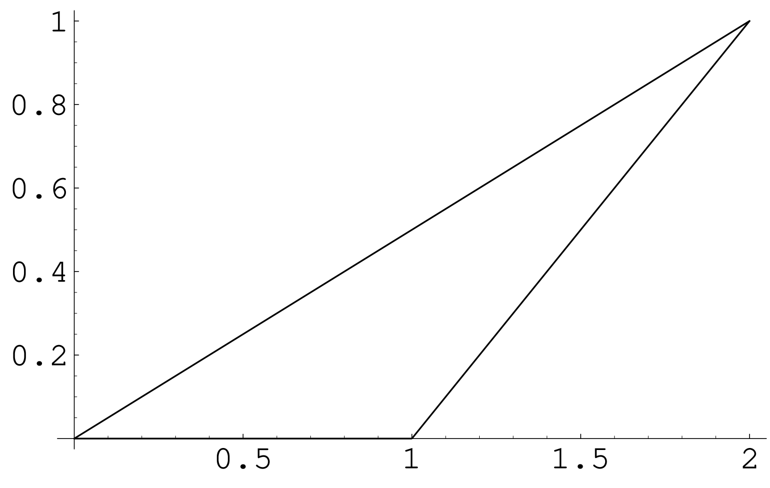

Figure 1: Range of U,V .

(ii) First solution - via Jacobians: The transformation $(X, Y) \to (U, V)$ is 2-1 and onto from A to B. On the set $x < y$, its inverse is given by $X = V$, $Y = U - V$; on the set $x > y$, its inverse is given by $X = U - V$, $Y = V$. These mappings are continuously differentiable on $B^* \equiv B \setminus \{(u, v) : v = u/2\} = B \setminus$ a null set. On B^* the Jacobian of the transformations are

$$\det \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = -1 \qquad \text{if } x < y, \qquad \det \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = 1 \qquad \text{if } x > y. \tag{1}$$

Thus by the usual transformation of densities formula, the joint density of (U, V) is obtained from $f_{X,Y}(x, y) = 1_{[0,1]}(x)1_{[0,1]}(y)$ as follows:

$$\begin{aligned} f_{U,V}(u, v) &= f_{X,Y}(x(u, v), y(u, v))|\det \frac{\partial(x, y)}{\partial(u, v)}|1_{[x(u,v)<y(u,v)]} \\ &\qquad + f_{X,Y}(x(u, v), y(u, v))|\det \frac{\partial(x, y)}{\partial(u, v)}|1_{[x(u,v)>y(u,v)]} \\ &= (1_{[0,1]}(v)1_{[0,1]}(u - v) \cdot 1 + 1_{[0,1]}(u - v)1_{[0,1]}(v) \cdot 1)1_{[v<u/2]} \\ &= 2 \cdot 1_B(u, v) \, . \end{aligned}$$

Thus the joint density of (U, V) is uniform on B. The random variables U and V are clearly *not* independent since the range of (U, V) is not a product set in R^2; moreover, the joint density of (U, V) does not factor into the product of its marginal densities. [The marginal densities are given by

$$f_U(u) = \int f_{U,V}(u, v)dv = \begin{cases} \int_0^{u/2} 2dv = u, & u \in [0, 1] \\ \int_{u-1}^{u/2} dv = 2 - u, & u \in (1, 2] \end{cases}$$

and

$$f_V(v) = \int f_{U,V}(u, v)du = \int_{2v}^{v+1} du = (1 - v)1_{[0,1]}(v).]$$

Second solution by direction calculation of the joint distribution function: Note that we can write

$$\begin{aligned} &P(U \le u, V > v) \\ &= P(X + Y \le u, X \wedge Y > v) = P(X + Y \le u, X > v, Y > v) \\ &= \begin{cases} \frac{1}{2}(u - 2v)^2 & \text{if } 0 < u < 1, \ 2v < u, \\ \frac{1}{2}(u - 2v)^2 & \text{if } 1 < u < 2, \ u - 1 < v < u/2, \\ \{(1 - v)^2 - (1/2)(2 - u)^2\} & \text{if } 1 < u < 2, \ v < u - 1 \end{cases} . \end{aligned}$$

Computing $-(\partial^2/\partial u \partial v)P(U \le u, V > v)$ on each of these pieces separately again yields $f_{U,V}(u,v) = 21_B(u,v)$.

2. Prove part (ii) of Proposition 1.1: There exists a minimal field, $\sigma-$field, and monotone class generated by any class of subsets of Ω.
Solution: Let $\mathcal{C}$ be a class of subsets of Ω. By part (i) of Proposition 1.1, the intersection of any family of fields, $\sigma-$fields, or monotone classes, are again fields, $\sigma-$fields, or monotone classes, respectively. Consider the collection $\{\mathcal{A}_\lambda\}$, of all fields containing $\mathcal{C}$. This collection is non-empty since it contains the field 2^Ω containing all subsets of Ω. Thus by Proposition 1.1(i) $\cap_\lambda \mathcal{A}_\lambda$ is again a field. This is clearly the minimal field containing $\mathcal{C}$ since $\cap_\lambda \mathcal{A}_\lambda \subset \mathcal{A}_{\lambda'}$ for each particular field $\mathcal{A}_{\lambda'}$ containing $\mathcal{C}$. The proofs for $\sigma-$fields and monotone classes is exactly the same.

3. (a) Suppose that $\{\mathcal{A}_n\}$ is an increasing sequence of fields, i.e. $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ for all $n \ge 1$. Show that $\cup_{n=1}^\infty \mathcal{A}_n$ is a field. (b) Suppose that the $\mathcal{A}_n$ of (a) are $\sigma-$ fields. Show by constructing a counterexample that $\cup_{n=1}^\infty \mathcal{A}_n$ need not be a $\sigma-$field.
Solution: (a) Let $\mathcal{A} = \cup_{n=1}^\infty \mathcal{A}_n$.
(i) If $A \in \mathcal{A}$, then $A \in \mathcal{A}_n$ for some n, so $A^c \in \mathcal{A}_n$ (since $\mathcal{A}_n$ is a field), which implies $A^c \in \mathcal{A}$.
(ii) If $A, B \in \mathcal{A}$, then $A \in \mathcal{A}_m$, $B \in \mathcal{A}_n$ for some m, n. Suppose that $m \le n$. Then $A \in \mathcal{A}_n$ since $\mathcal{A}_m \subset \mathcal{A}_{m+1} \subset \ldots \subset \mathcal{A}_n$. Hence $A \cup B \in \mathcal{A}_n$ (since $\mathcal{A}_n$ is a field). Hence $A \cup B \in \cup_{n=1}^\infty \mathcal{A}_n = \mathcal{A}$.
By (i) and (ii) $\mathcal{A}$ is a field.
(b) Let $\mathcal{A}_1 = \sigma(\{\emptyset, [0,1]\})$, $\mathcal{A}_2 = \sigma(\mathcal{A}_1 \cup \{[0,1/2]\})$, $\mathcal{A}_3 = \sigma(\mathcal{A}_2 \cup \{[0,1-1/3]\})$, ..., $\mathcal{A}_n = \sigma(\mathcal{A}_{n-1} \cup \{[0,1-1/n]\})$, ... Then each $\mathcal{A}_n$ is a finite $\sigma-$field with $\sharp(\mathcal{A}_n) = 2^n$; clearly $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ by construction. Consider the sets $A_n = [0, 1-1/n] \in \mathcal{A}_n$ for $n \ge 2$. Then $A = \cup_{n=2}^\infty A_n = [0,1) \notin \cup_{n=1}^\infty \mathcal{A}_n$. [If $A = [0,1) \in \cup_{n=1}^\infty \mathcal{A}_n$, then $A \in \mathcal{A}_{n_0}$ for some n_0, but this fails. All the sets in $\cup_{n=1}^\infty \mathcal{A}_n$ are closed on the right.]

4. Let μ be a finite measure on R, and let $G(x) = \mu((-\infty, x])$. Show that
$$\int [G(x+c) - G(x)]dx = c\mu(R).$$
[Hint: use Fubini's theorem.]
Solution: Note that $G(x+c) - G(x) = \int 1_{(x,x+c]}(y) d\mu(y)$, so that by

using the Tonelli part of the Tonelli-Fubini theorem 3.??,

$$\begin{aligned}
\int [G(x+c) - G(x)]dx &= \int \left\{ \int 1_{(x,x+c]}(y) d\mu(y) \right\} dx \\
&= \int \left\{ \int 1_{(x,x+c]}(y) dx \right\} d\mu(y) \\
&= \int \left\{ \int 1_{[y-c,y)}(x) dx \right\} d\mu(y) \\
&= \int c d\mu(y) = c\mu(R) .
\end{aligned}$$

5. Let $\mathcal{X} = (0,1)$, $\mathcal{Y} = (1,\infty)$, both equipped with the Borel sets and Lebesgue measure. Let $f(x,y) = e^{-xy} - 2e^{-2xy}$. Show that:
(a) $\int_0^1 (\int_1^\infty f(x,y)dy)dx = \int_0^1 x^{-1}(e^{-x} - e^{-2x})dx$ exists and is > 0.
(b) $\int_1^\infty (\int_0^1 f(x,y)dx)dy = \int_1^\infty y^{-1}(e^{-2y} - e^{-y})dy$ exists and is < 0.
(c) Why does Fubini's theorem fail here?
Solution: (a) The first inner integral is

$$\begin{aligned}
\int_1^\infty f(x,y)dy &= \int_1^\infty \{e^{-xy} - 2e^{-2xy}\}dy \\
&= (-x^{-1}e^{-xy} + x^{-1}e^{-2xy}|_{y=1}^\infty \\
&= x^{-1}(e^{-x} - e^{-2x}) \equiv g(x) .
\end{aligned}$$

Note that $g(x) \geq 0$ with strict inequality for $x > 0$ since $e^{-x} \geq e^{-2x}$ with equality only at $x = 0$, and $g(x) \leq x^{-1}(1 - x/2 - (1-2x)) = 3/2$, since $e^{-x} \leq 1 - x/2$ for $0 \leq x \leq 1$ and $e^{-2x} \geq 1 - 2x$. Thus $\int_0^1 g(x)d > 0$ and

$$\begin{aligned}
\int_0^1 (\int_1^\infty f(x,y)dy)dx &= \int_0^1 x^{-1}(e^{-x} - e^{-2x})dx \\
&\leq \int_0^1 (3/2)dx = 3/2 < \infty .
\end{aligned}$$

In fact, with $Ei(z) \equiv -\int_{-z}^\infty t^{-1}e^{-t}dt$,

$$\int_0^1 x^{-1}(e^{-x} - e^{-2x})dx$$

$$
\begin{aligned}
&= \int_0^\infty x^{-1}(e^{-x} - e^{-2x})dx - \int_1^\infty x^{-1}(e^{-x} - e^{-2x})dx \\
&= \log(2) + Ei(-1) - Ei(-2) = .522664...
\end{aligned}
$$

(via Mathematica).
(b) The second inner integral is

$$
\begin{aligned}
\int_0^1 f(x,y)dx = \int_0^1 \{e^{-xy} - 2e^{-2xy}\}dx &= (-y^{-1}e^{-xy} + y^{-1}e^{-2xy}|_{x=0}^1 \\
&= y^{-1}(e^{-2y} - e^{-y}) \equiv h(y) \, .
\end{aligned}
$$

Note that $h(y) < 0$, $1 \le y < \infty$, with strict inequality since $e^{-2y} < e^{-y}$. Further, $h(y) \ge e^{-2y}$. Thus $\int_1^\infty h(y)dy < 0$ and

$$
\begin{aligned}
\int_1^\infty (\int_0^1 f(x,y)dx)dy &= \int_1^\infty y^{-1}(e^{-2y} - e^{-y})dy \\
&\ge \int_1^\infty -e^{-y}dy = e^{-y}|_1^\infty = -e^{-1} \, .
\end{aligned}
$$

In fact

$$
\begin{aligned}
&\int_1^\infty y^{-1}(e^{-2y} - e^{-y})dy \\
&= -Ei(-2) + Ei(-1) = -.170483\ldots
\end{aligned}
$$

(via Mathematica).
(c) Since the two iterated integrals differ, it follows from the Fubini-Tonelli theorem that

$$
\int_0^1 \int_1^\infty |f(x,y)|dydx = \infty \, ;
$$

if this integral (which can be computed by either of the iterated integrals) were finite, then we would have $f \in L_1(dx \times dy)$, and by the Fubini part of the Fubini-Tonelli theorem the two iterated integrals would agree. It is instructive in this case to compute the above integral explicitly to see where the divergence happens. Note that

$$
|f(x,y)| = \begin{cases} e^{-xy} - 2e^{-2xy} & \text{if } \ xy \ge \log 2 \\ 2e^{-2xy} - e^{-xy} & \text{if } \ xy \le \log 2 \end{cases} \, .
$$

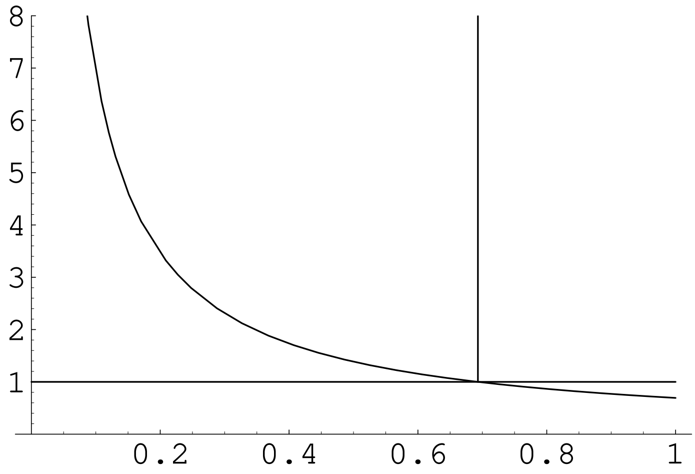

Figure 2: Three regions for integration of $|f|$.

Hence

$$\begin{aligned}
\int_0^1 \int_1^\infty |f(x,y)| dx dy &= \int\int_{xy>\log 2} (e^{-xy} - 2e^{-2xy}) dx dy \\
&\quad + \int\int_{xy<\log 2} (2e^{-2xy} - e^{-xy}) dx dy \\
&= \int_0^{\log 2} \left(\int_1^{\log 2/x} (2e^{-2xy} - e^{-xy}) dy \right) dx \\
&\quad + \int_0^{\log 2} \left(\int_{\log 2/x}^\infty (e^{-xy} - 2e^{-2xy}) dy \right) dx \\
&\quad + \int_{\log 2}^1 \left(\int_1^\infty (e^{-xy} - 2e^{-2xy}) dy \right) dx \\
&= \int_0^{\log 2} \left(\frac{1}{4x} + \frac{1}{x}(e^{-2x} - e^{-x}) \right) dx \\
&\quad + \int_0^{\log 2} \frac{1}{4x}\, dx
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{\log 2}^{1} \frac{1}{x}(e^{-x} - e^{-2x})\,dx \\
&= \infty + \infty + \text{something finite} \\
&= \infty\,.
\end{aligned}
$$

6. Lehmann, TPE, problem 1.3, page 62.
Solution: (i) First solution – via the Cauchy-Schwarz inequality: First recall the Cauchy-Schwarz inequality in R^n: if $u, v \in R^n$, then $(u'v)^2 \leq (u'u)(v'v)$ with equality iff $u = cv$ for some real number c. Now extend this as follows: if Σ is positive definite and $x, y \in R^n$, then

$$(x'y)^2 \;=\; (\Sigma^{1/2}x)'(\Sigma^{-1/2}y) \leq (x'\Sigma x)(y'\Sigma^{-1}y)$$

with equality iff $\Sigma^{1/2}x = c\Sigma^{-1/2}y$; i.e. iff $x = c\Sigma^{-1}y$.

Now consider X, a random vector in R^n, with $E(X) = \mathbf{1}\theta$ and $Cov(X, X) = E[(X - E(X))(X - E(X))'] = \Sigma$, where $\mathbf{1} = (1, \ldots, 1)' \in R^n$. A linear estimator $\alpha'X = \alpha_1 X_1 + \ldots + \alpha_n X_n$ is unbiased for θ iff $\theta = E(\alpha'X) = \alpha'E(X) = (\alpha'\mathbf{1})\theta$ for all θ; i.e., iff $\alpha'\mathbf{1} = 1$. The variance of $\alpha'X$ is $Var(\alpha'X) = \alpha'\Sigma\alpha$. To find the best such estimator, we must find

$$\min\{\alpha'\Sigma\alpha : \alpha'\mathbf{1} = 1\}.$$

But by the Cauchy-Schwarz inequality, if $\alpha'\mathbf{1} = 1$, then

$$\alpha'\Sigma\alpha \geq 1/(\mathbf{1}'\Sigma^{-1}\mathbf{1})$$

with equality iff $\alpha = c\Sigma^{-1}\mathbf{1}$. The condition $\alpha'\mathbf{1} = \mathbf{1}$ then implies that $c = 1/(\mathbf{1}'\Sigma^{-1}\mathbf{1})$, so the optimal α is $\alpha_0 \equiv \Sigma^{-1}\mathbf{1}/(\mathbf{1}'\Sigma^{-1}\mathbf{1})$, and the optimal linear unbiased estmator is $\alpha_0'X = (\mathbf{1}'\Sigma^{-1}X)/(\mathbf{1}'\Sigma^{-1}\mathbf{1})$ whose variance is $Var(\alpha_0'X) = 1/(\mathbf{1}'\Sigma^{-1}\mathbf{1})$.
The solutions to (i) and (ii) now follow:
(i) The inverse of the matrix diag$(1/a_i)$ is just diag(a_i). This implies that $\alpha_0'X = (\sum_1^n a_i X_i)/(\sum_1^n a_i)$ and $Var(\alpha_0'X) = \sigma^2/\sum a_i$.
(ii) The inverse of the matrix with 1 on the diagonal and ρ off the diagonal is of the form a in the diagonal entries and b in the off-diagonal entries for some a, b. Hence $\Sigma^{-1}\mathbf{1} = \sigma^{-2}(a + (n-1)b)\mathbf{1}$, which leads

to $\mathbf{1}'\Sigma^{-1}X = \sigma^{-2}(a+(n-1)b)(X_1+\ldots+X_n)$, and $\mathbf{1}'\Sigma^{-1}\mathbf{1} = \sigma^{-2}(a+(n-1)b)n$. Hence we find that $\alpha_0'X = \sum_1^n X_i/n$. But $\Sigma\mathbf{1} = \sigma^2(1+(n-1)\rho)\mathbf{1}$, so $\mathbf{1} = \sigma^2(1+(n-1)\rho)\Sigma^{-1}\mathbf{1} = (1+(n-1)\rho)(a+(n-1)b)\mathbf{1}$, and hence $[a+(n-1)b] = [1+(n-1)\rho]^{-1}$. Therefore

$$Var(\alpha_0'X) \;=\; \frac{\sigma^2}{n}[1+(n-1)\rho]\begin{cases} > \sigma^2/n & \text{if } \rho > 0 \\ < \sigma^2/n & \text{if } -1/(n-1) \leq \rho < 0 \end{cases}.$$

[Note that if $\rho < -1/(n-1)$, the matrix Σ of this form *is not a covariance matrix*!