

Statistics 581, Problem Set 9

Wellner; 11/22/2000

Reading: Chapter 4, Sections 1-2;

Ferguson, ACLST, Chapter 20, pages 133-139; Chapter 22, pages 144-150;

Lehmann and Casella, Chapter 6, especially section 6.5, pages 461-468.

Due: Wednesday, November 29, 2000.

- Lehmann and Casella, problem 6.3.1, page 501.
 - Lehmann and Casella, problem 6.3.2, page 501.
 - Lehmann and Casella, problem 6.3.5, page 501.
- Suppose that $(Y|Z) \sim \text{Poisson}(\lambda e^{\gamma Z})$, and $Z \sim \text{Bernoulli}(\eta)$, and $\theta = (\lambda, \gamma, \eta)$. Let $X = (Y, Z)$, and suppose that we observe X_1, \dots, X_n i.i.d. as X .
 - Find the score equations for estimation of θ .
 - Give conditions on the data $X_1, \dots, X_n = (Y_1, Z_1), \dots, (Y_n, Z_n)$ guaranteeing that the score equations have a unique solution which maximizes the likelihood. Call the resulting estimators $\hat{\theta}_n = (\hat{\lambda}_n, \hat{\gamma}_n, \hat{\eta}_n)$.
 - What does theorem 4.1.5 (Chapter 4, page 4), say about the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta_0)$ when the distribution of the data is given by P_{θ_0} .
 - Suppose that $\theta_1 \neq \theta_0$ is the “true” value of the parameter θ , and we consider the likelihood ratio $L_n(\theta_1)/L_n(\theta_0)$ where $L_n(\theta) \equiv \prod_{i=1}^n p_{\theta}(X_i)$. Show that $n^{-1} \log(L_n(\theta_1)/L_n(\theta_0)) \rightarrow_p$ some constant, and identify the constant explicitly in terms of θ_1, θ_0 .
- For the same set-up as in problem 1, consider taking a “profile likelihood” approach to the estimation of γ as follows:
 - Let $l_n(\theta) = l_n(\gamma, \lambda, \eta)$: consider first maximizing this as a function of λ and η for each fixed value of γ to find

$$(\hat{\lambda}(\gamma), \hat{\eta}(\gamma)) \equiv \operatorname{argmax}_{(\lambda, \eta)} l_n(\lambda, \gamma, \eta).$$

Compute the maximizer $(\hat{\lambda}(\gamma), \hat{\eta}(\gamma))$ as explicitly as possible, and then form the “profile log-likelihood” $l_n^{\text{profile}}(\gamma)$ defined by

$$l_n^{\text{profile}}(\gamma) \equiv l_n(\hat{\lambda}(\gamma), \gamma, \hat{\eta}(\gamma)).$$

- Now maximize $l_n^{\text{profile}}(\gamma)$ with respect to γ . Find the resulting “profile likelihood” score equation for γ .
 - Does the equation you derived in (b) follow from the original score equations?
 - Does the “profile score function” which appears in (b) correspond to or relate to the efficient score for γ in any way?
- Optional bonus problem:** Lehmann and Casella, TPE, problem 6.3.22, page 503, reworded as follows. (In other words, prove (vi) of theorem 1.5, page 5, chapter 4 notes). Suppose that X_1, \dots, X_n are i.i.d. with density p_{θ} , $\theta \in \Theta \subset R^k$, satisfying the hypotheses of theorem 4.1, page 429 (the Cramér conditions given in (A) - (D)

on page 429). Show that the following Local Asymptotic Normality (LAN) result holds for the (local) log-likelihood ratios: with

$$L_n(\theta) \equiv \log\left(\prod_{i=1}^n p_\theta(X_i)\right) = \sum_{i=1}^n \log p_\theta(X_i),$$

for a fixed $\theta_0 \in \Theta$,

$$\begin{aligned} L_n(\theta_0 + n^{-1/2}\underline{t}) - L_n(\theta_0) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \underline{t}^T \dot{l}_\theta(X_i) - \frac{1}{2} \underline{t}^T I(\theta_0) \underline{t} + o_p(1) \\ &\rightarrow_d N(0, \underline{t}^T I(\theta_0) \underline{t}) - \frac{1}{2} \underline{t}^T I(\theta_0) \underline{t} =_d N\left(-\frac{1}{2} \underline{t}^T I(\theta_0) \underline{t}, \underline{t}^T I(\theta_0) \underline{t}\right) \end{aligned}$$

under P_{θ_0} where $\sigma^2 \equiv \underline{t}^T I(\theta_0) \underline{t}$.