Statistics 581, Problem Set 8

Wellner; 11/15/2000

Reading: Chapter 3, Section 2;
Ferguson, ACLST, Chapter 19, pages 126-132, Chapter 20, pages 133-134;
Lehmann and Casella, pages 113-129, and 439- 443;
begin reading Chapter 4 (to be handed out on Friday 11/17).
Due: Wednesday, November 22, 2000.
Reminder: No class on Monday, November 20, 2000.

1. Suppose that $(Y|Z) \sim \text{Poisson}(\lambda e^{\gamma Z})$, and $Z \sim G_\eta$ on R with density g_η with respect to some dominating measure μ. You may assume that

$$a(z) \equiv (\partial/\partial\eta) \log g_\eta(z)$$

exists and $E\{a^2(Z)\} < \infty$. Thus Z is a "covariate" or "predictor variable", γ is a "regression parameter" which affects the intensity of the (conditionally) Poisson variable Y, and $\theta = (\lambda, \gamma, \eta)$.
(a) Find the information matrix for θ. What does the structure of this matrix say about the effect of η being known or unknown about the estimation of λ and γ?
(b) Find the information and information bound for γ if the parameter λ is known.
(c) Find the efficient score function and the efficient influence function for estimation of γ when λ is known.
(d) Find the information and information bound for γ if the parameter λ is unknown, $I_{\gamma\gamma\cdot\lambda}$.
(e) Find the efficient score function and the efficient influence function for estimation of γ when λ is unknown.
(f) In the case when $Z \sim \text{Bernoulli}(\eta)$, compute the ratio of the information when λ is unknown, to the information when λ is known as a function of γ and of η.

2. For the same set-up as in problem 1, but with the distribution G of Z assumed to be known (so η is known), consider the two parameters $q_1(\theta) = E_\theta(Y) = \nu_1(P_\theta)$ and $q_2(\theta) = P_\theta(Y \le y_0) = \nu_2(P_\theta)$ for a fixed integer y_0.
(a) Find the information bounds for estimation of $q_1(\theta)$ and $q_2(\theta)$.
(b) Find the efficient influence functions $\tilde{l}_{\nu_1}$ and $\tilde{l}_{\nu_2}$ for estimation of q_1 and q_2.
(c) If $(Y_1, Z_1), \ldots, (Y_n, Z_n)$ are i.i.d. as (Y, Z), find the influence functions ψ_1 and ψ_2 and the asymptotic variances $V_1^2(\theta)$ and $V_2^2(\theta)$ of the natural empirical estimators of q_1 and q_2 respectively.
(d) Show that $\Pi(\psi_i|\dot{\mathcal{P}}) = \tilde{l}_{\nu_i}$ for $i = 1, 2$.

3. **Optional bonus problem:** Suppose that $X \sim \text{Gamma}(\alpha, \beta)$; i.e. X has density p_θ given by

$$p_\theta(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x) 1_{(0,\infty)}(x), \quad \theta = (\alpha, \beta) \in (0, \infty) \times (0, \infty) \equiv \Theta\,.$$

Consider estimation of : A. $q_A(\theta) \equiv E_\theta X$. B. $q_B(\theta) \equiv F_\theta(x_0)$ for a fixed x_0; here $P_\theta(x) \equiv P_\theta(X \le x)$.

(i) Compute $I(\theta) = I(\alpha, \beta)$; compare Lehmann & Casella page ??
(ii) Compute $q_A(\theta)$, $q_B(\theta)$, $\dot{q}_A(\theta)$, and $\dot{q}_B(\theta)$.
(iii) Find the efficient influence functions for estimation of q_A and q_B.
(iv) Compare the efficient influence functions you find in (iii) with the influence functions ψ_A and ψ_B of the natural nonparametric estimators $\overline{X}_n$ and $\mathbb{F}_n(x_0)$ respectively; in particular, show that $\psi_A \in \dot{\mathcal{P}}$, while $\psi_B \notin \dot{\mathcal{P}}$.