Statistics 581, Problem Set 4

Wellner; 10/16/00

Reading: Chapter 2, sections 3 and 4.
Due: Wednesday, October 23, 2000.

1. Suppose that $\underline{N}_n \sim \text{Mult}_k(n, \underline{p}_n)$ where $\underline{p}_n = \underline{p_0} + n^{-1/2}\underline{c}$. Thus $\underline{N}_n = \sum_{i=1}^n \underline{M}_{ni}$ where $\underline{M}_{n1}, \ldots, \underline{M}_{n1}$ are i.i.d. $\text{Mult}_k(1, \underline{p}_n)$.
   Show that with $\widehat{\underline{p}}_n \equiv \underline{N}_n/n$ we have
   $$\sqrt{n}(\widehat{\underline{p}}_n - \underline{p}_n) \rightarrow_d N_k(0, \Sigma)$$
   where $\Sigma = \text{diag}(\underline{p}_0) - \underline{p}_0\underline{p}_0$.
   Hint: Use the Cramér-Wold device and the Liapunov CLT.

2. Suppose that $\underline{N}_n \sim \text{Mult}_k(n, \underline{p})$, and let $\widehat{\underline{p}}_n = \underline{N}_n/n$ as before. Consider the square of the Hellinger distance,
   $$H^2(\widehat{\underline{p}}_n, \underline{p}_0) = \sum_{j=1}^k (\sqrt{\widehat{p}_j} - \sqrt{p_{0j}})^2 .$$
   (a) Show that if the null hypothesis $\underline{p} = \underline{p}_0$ holds, then $4nH^2(\widehat{\underline{p}}_n, \underline{p}_0) \rightarrow_d \chi^2_{k-1}$.
   (b) Show that if $\underline{p} \neq \underline{p}_0$ holds, then $4H^2(\widehat{\underline{p}}_n, \underline{p}_0) \rightarrow_p$ "something"> 0, and find "something".
   (c) Show that if $\underline{p} = \underline{p}_0 + n^{-1/2}\underline{c}$ holds, with $\underline{1}'\underline{c} = 0$, then $4nH^2(\widehat{\underline{p}}_n, \underline{p}_0) \rightarrow_d \chi^2_{k-1}(\delta)$ where $\delta = \sum_1^k c_i^2/p_{i0}$.

3. Lehmann & Casella (1998), problem 5.22, page 406, define a noncentral chi-square distribution as follows: $Y \sim \chi^2_r(\lambda)$ if $(Y|K = k) \sim \chi^2_{2k+r}$ and $K \sim Poisson(\lambda)$. If $X \sim N_r(\mu, I)$, what is the distribution of $X'X = |X|^2$
   (a) In the notation I have used in class.
   (b) In Lehmann's notation.
   (c) Is there any contradiction or false statement in Lehmann & Casella's problem 5.22? (Note the Poisson($\lambda/2$) in problem 5.23!).

4. Suppose that $X_1, X_2, \ldots$ are i.i.d. (μ, σ^2) with $\mu_4 < \infty$. Let $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$ and $S_n^2 = (n-1)^{-1}\sum_{i=1}^n (X_i - \overline{X}_n)^2$ be the sample mean and sample variance respectively.
   (a) Show that
   $$\sqrt{n}\begin{pmatrix} \overline{X}_n - \mu \\ S_n^2 - \sigma^2 \end{pmatrix} \rightarrow_d \underline{Z} \sim N_2(0, \Sigma)$$
   where
   $$\begin{pmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{pmatrix}.$$
   (b) Suppose $\mu \neq 0$. Use (a) to find the limiting distribution of the sample *coefficient of variation* $C_n \equiv S_n/\overline{X}_n$; i.e. show that $\sqrt{n}(C_n - c) \rightarrow_d N(0, V^2)$ with $c \equiv \sigma/\mu$ and find V^2.

5. **Optional bonus problem:** (a) (From Ferguson, *A Course in Large Sample Theory*, page 65.) In a multinomial experiment with sample size $n = 100$ and 3 cells with null hypothesis $H_0 : \underline{p}_0 = (.25, .5, .25)$, what is the approximate power at the alternative $\underline{p} = (.2, .6, .2)$ when the level of significance is $\alpha = .05$? $\alpha = .01$? How large a sample size is need to achieve power 0.9 at this alternative when $\alpha = .05$? $\alpha = .01$?
(b) Show that if:
(i) $X \sim N_k(\delta, \Sigma)$ where Σ is of rank $r \leq k$,
(ii) Σ is a projection matrix (so $\Sigma^2 = \Sigma$).
(iii) $\Sigma\delta = \delta$,
then $X^T X \sim \chi^2_r(\delta^T \delta)$.