Statistics 582, Problem Set 2

Wellner; 10/4/00

Reading:
Due: Wednesday, October 11, 2000.

1. Ferguson, ACILST, #2, page 6:
(a) Suppose that $X_n \sim \text{Uniform}\{1/n, 2/n, \ldots, n/n\}$ Show that $X_n \rightarrow_d X \sim \text{Uniform}(0,1)$. Does $X_n \rightarrow_p X$?
(b) Suppose that $Y_n = \sum_{k=1}^{n}(k/n)1_{[k-1/n,k/n)}(U)$ where $U \sim \text{Uniform}[0,1]$. Show that $Y_n \sim$Uniform$\{1/n, 2/n, \ldots, n/n\}$, and $Y_n \rightarrow_p U$.

2. Ferguson, ACILST, #6, page 7. (This is known as the Polya-Cantelli lemma; see Chapter 2, Proposition 2.11, page 10.

3. Suppose that $U \sim \text{Uniform}(0,1)$, $\alpha > 0$, and
$$X_n \equiv (n^\alpha / \log(n+1))1_{[0,1/n^\alpha]}(U)\,.$$
(a) Show that $X_n \rightarrow_{a.s.} 0$ and $E(X_n) \rightarrow E(0) = 0$.

(b) Can you find a random variable Y with $|X_n| \le Y$ for all n with $E(Y) < \infty$ for any α?
(c) For what values of α does the uniform integrability condition
$$\limsup_{n\rightarrow\infty} E\{|X_n| 1_{[|X_n|\ge M]}\} \rightarrow 0 \qquad \text{as} \;\; M \rightarrow \infty$$
hold?

4. Suppose that $X \sim \text{Uniform}(0,1)$ and $Y = 3X$.
(a) Find the joint distribution function $F(x,y) = F_{X,Y}(x,y)$ of (X,Y).
(b) Is F a continuous function?
(c) Is the probability measure P corresponding to F absolutely continuous with respect to Lebesgue measure μ on R^2?

5. (a) Lehmann and Casella, #3.5, page 64.
(b) Lehmann and Casella, #3.6, page 64.
(c) Lehmann and Casella, #3.7, page 64.

6. Suppose that $X \sim F$ on $R^+ \equiv [0, \infty)$, $Y \sim G$ on R^+, and X and Y are independent random variables. Let $Z = \min\{X, Y\} = X \wedge Y$ and $\Delta = 1\{X \leq Y\}$. (This is *right-censored data*: if we view X as a survival time, and Y as a censoring time, then $Z = X$ when $X \leq Y$, but $Z = Y$ when $X > Y$.)
(a) Find the joint distribution of (Z, Δ).
(b) If $X \sim$ Exponential(λ) and $Y \sim$ Exponential(μ), show that Z and Δ are independent.
[Hint: for (a), compute $P(Z \leq z, \Delta = 1)$ and $P(Z \leq z, \Delta = 0)$.]

7. **Optional Bonus Problem:**
(a) If X is a random variable on a probability space $(\Omega, \mathcal{A}, P)$, show that the distribution function $F = F_X$ of X defined by $F(x) = P(X \leq x)$ is right-continuous.
(b) Show that the discontinuity set $D_F \equiv \{x \in R : F \text{ is discontinuous at } x\}$ of F is countable. Hint: consider the intervals $(F(x-), F(x)]$.
(c) Define the inverse F^{-1} of F by $F^{-1}(u) \equiv \inf\{x : F(x) \geq u\}$ for $0 < u < 1$. Show that F^{-1} is left-continuous.
(d) Show that the discontinuity set $D_{F^{-1}} = \{u \in (0, 1) : F^{-1} \text{is discontinuous at } u\}$ is countable.