Notes on Convergence in Law of Maxima

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1. Introduction

For the basics of convergence in distribution of maxima of independent random variables, see Van der Vaart (1998), section 21.4, pages 312-314, and Ferguson (1996), chapter 14, pages 94-100. For further more recent results in this vein, see Engelke et al. (2015) and Kabluchko (2011a, 2011b, 2014).

2. Example 1: maxima of i.i.d. standard normal

Suppose that $X_1, \ldots, X_n$ are i.i.d. with d.f. $F = \Phi$, the standard normal distribution. Then as discussed by van der Vaart (1998), section 21.4, pages 312-314, $X_{(n)} \equiv M_n \equiv \max_{1 \le i \le n} X_i$ satisfies

$$G_n \equiv b_n(M_n - a_n) \to_d G \sim Ev \tag{2.1}$$

where $Ev(x) = \exp(-\exp(-x))$,

$$\begin{aligned} b_n &\equiv \sqrt{2\log n}, \\ a_n &\equiv \sqrt{2\log n} - \frac{1}{2}\frac{\log\log n + \log(4\pi)}{\sqrt{2\log n}} \end{aligned}$$

Moreover, the density $f_{G_n}(x) \to Ev'(x)$ and $d_{TV}(P_{G_n}, P_G) \to 0$. Figure 1 illustrates the convergence in (2.1) and the claimed convergence of densities is illustrated in Figure 2. The rate of convergence in (2.1) is $O(1/\log n)$; see Hall (1979) and Resnick (1987), page 121.

3. Example 2: maxima of i.i.d. standard exponentials

Now suppose that $X_1, \ldots, X_n$ are i.i.d. with distribution function F given by $1 - F(x) = \exp(-x)$. Again let $M_n \equiv X_{(n)}$. Then it is easily seen that

$$G_n \equiv M_n - \log n \to_d G \sim Ev \tag{3.1}$$

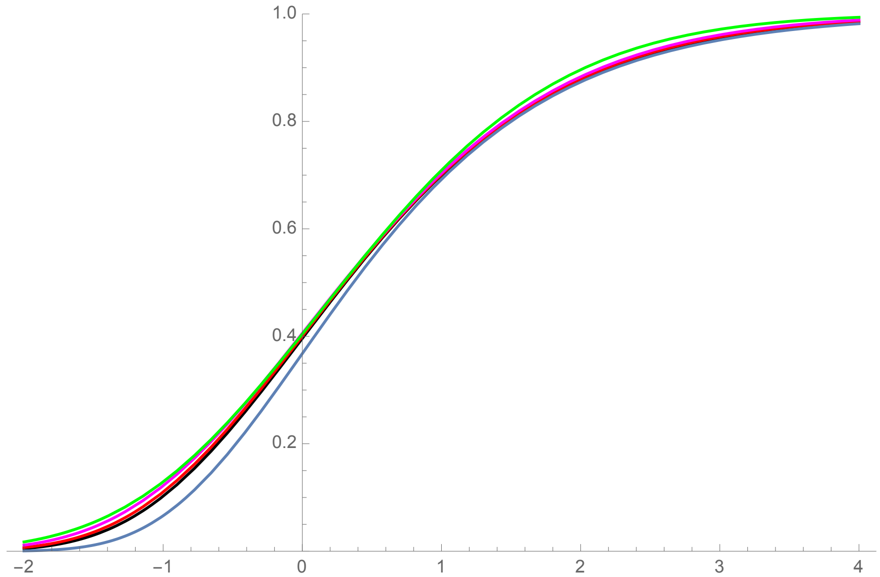

Figure 1: Blue, limit distribution Ev; Green, $n = 10$; Magenta, $n = 100$; Red, $n = 1000$; Black, $n = 10^4$

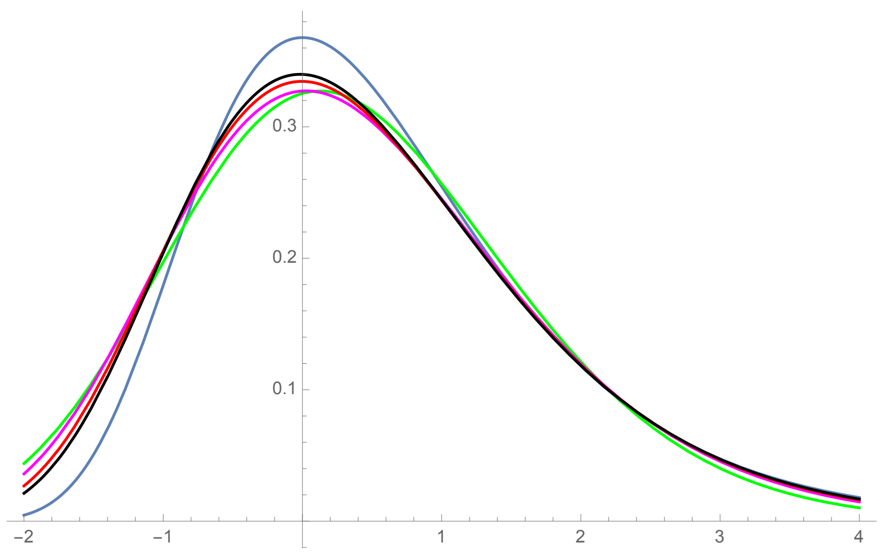

Figure 2: Blue, limit distribution Ev; Green, $n = 10$; Magenta, $n = 100$; Red, $n = 1000$; Black, $n = 10^4$

This follows since

$$P(M_n - \log n \leq x) \quad = \quad P(X_{(n)} \leq x + \log n) = (1 - \exp(-(x + \log n)))^n$$

$$= \left(1 - \frac{e^{-x}}{n}\right)^n \to \exp(-\exp(-x)) = Ev(x)$$

for every x. Furthermore,

$$\begin{aligned} f_{G_n}(x) &= n\left(1 - \exp(-(x + \log n))\right)^{n-1} \exp(-(x + \log n)) \\ &\to \exp(-\exp(-x))\exp(-x) = Ev'(x) \end{aligned}$$

for every x, and hence by Scheffé's theorem, $d_{TV}(P_{G_n}, P_G) \to 0$. Figure 3 illustrates the convergence in (3.1) and the claimed convergence of densities is illustrated in Figure 4. The rate of convergence in both cases is n^{-1}; see Hall & W (1979).

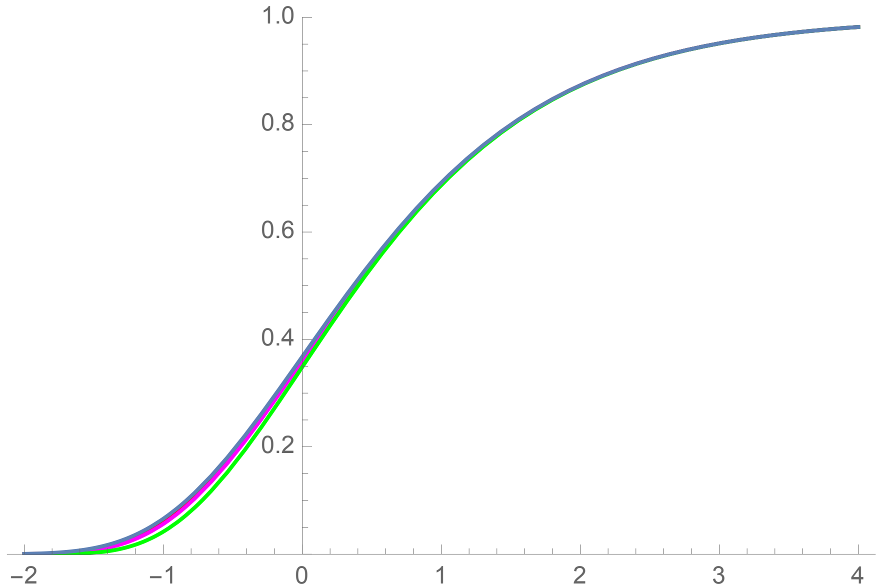

Figure 3: Blue, limit distribution Ev; Green, $n = 10$; Magenta, $n = 25$; Red, $n = 50$

4. Example 3: supremum of a standard kernel estimator

Let $\hat{f}_n$ be the kernel estimator of a density f on $[0, 1]$ based on a kernel w and the bandwidth $h_n = n^{-\delta}$ with $1/5 < \delta < 1/2$. Bickel and Rosenblatt (1973) show (under hypotheses specified in their paper) that

$$\tilde{M}_n \equiv \sup_{0<t<1} \frac{\sqrt{nh_n}|\hat{f}_n(t) - f(t)|}{\sqrt{f(t)}}$$

satisfies the following extreme value convergence:

$$\sqrt{2\delta \log n}\left(\frac{\tilde{M}_n}{\lambda(w)} - d_n\right) \to_d Ev^2$$

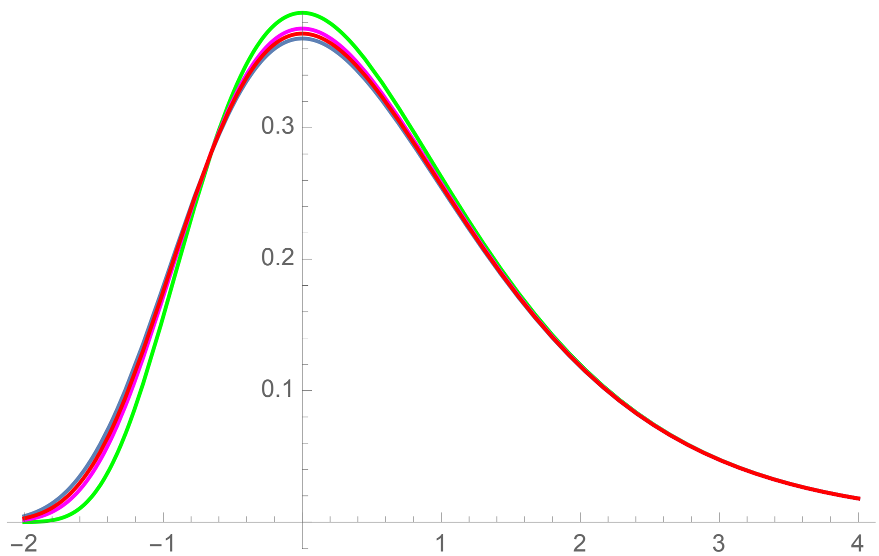

Figure 4: Blue, limit distribution Ev; Green, $n = 10$; Magenta, $n = 25$; Red, $n = 50$

where $Ev^2(x) = \exp(-2\exp(-x))$ and where

$$\lambda(w) \equiv \int w^2(t)dt,$$

$$K_1(w) \equiv \frac{w^2(A) + w^2(-A)}{2\lambda(w)},$$

$$K_2(w) \equiv \frac{1}{2\lambda(w)} \int \{w'(t)\}^2 dt,$$

$$d_n = \begin{cases} (2\delta \log n)^{1/2} + \frac{1}{(2\delta \log n)^{1/2}} \left\{ \frac{K_1(w)}{\sqrt{\pi}} - \frac{1}{2}(\log(\delta \log n)) \right\}, & \text{if } K_1(w) > 0, \\ \sqrt{2\delta \log n} + \frac{\log[K_2(w)/(2\pi)]}{\sqrt{2\delta \log n}}, & \text{otherwise.} \end{cases}$$

5. Example 4.

Let $\mathbb{G}_n$ be the empirical distribution function of i.i.d. uniform$(0,1)$ random variables and let $\mathbb{Z}_n(t) \equiv \sqrt{n}(\mathbb{G}_n(t) - t)/\sqrt{t(1-t)}$ for $0 < t < 1$. Then Jaeschke and Eicker showed that $\|\mathbb{Z}_n\|_\infty \equiv \sup_{0<t<1} |\mathbb{Z}_n(t)|$ satisfies

$$b_n \left(\|\mathbb{Z}_n\|_\infty - c_n/b_n \right) \to_d Ev^4$$

where

$$b_n \equiv \sqrt{2 \log \log n}, \qquad c_n \equiv 2 \log \log n + 2^{-1} \left(\log \log \log n - \log(4\pi) \right).$$

Here $Ev^4(x) = \exp(-4\exp(-x))$. See Shorack & W (1986), page 600.

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