Statistics 523, Problem Set 7, Solutions

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1. Use a reflection principle to show that for $0 \le y \le x$

$$P(\sup_{0\le s\le t} \mathbb{S}(s) \ge x,\ \mathbb{S}(t) \le y) = P(\mathbb{S}(t) \ge 2x - y)\,,$$

and use this to show that the joint density of $M^+ \equiv \sup_{0\le s\le t} \mathbb{S}(s), \mathbb{S}(t)$ is given by

$$f(x,y) = \sqrt{\frac{2}{\pi t^3}}(2x-y)\exp\left(-\frac{(2x-y)^2}{2t}\right) \qquad \text{for } \ 0 \le y \le x\,.$$

Solution: Note that by reflection

$$\begin{aligned}
P(\sup_{0\le s\le t} \mathbb{S}(s) \ge x,\ \mathbb{S}(t) \le y) &= P(\tau_x \le t, \mathbb{S}(\tau_x + t - \tau_x) - \mathbb{S}(\tau_x) \le y - x) \\
&= P(\tau_x \le t, \mathbb{S}(\tau_x + t - \tau_x) - \mathbb{S}(\tau_x) \ge x - y) \\
&= P(\tau_x \le t, \mathbb{S}(t) \ge 2x - y) \\
&= P(\mathbb{S}(t) \ge 2x - y) \\
&= 1 - \Phi\left(\frac{2x-y}{\sqrt{t}}\right)
\end{aligned}$$

since $[\mathbb{S}(t) \ge 2x - y] \subset [\tau_x \le t]$. Thus we compute the joint density f of $M^+ \equiv \sup_{0\le s\le t} \mathbb{S}(s), \mathbb{S}(t)$ as

$$\begin{aligned}
f(x,y) &= -\frac{\partial^2}{\partial x \partial y} P(\sup_{0\le s\le t} \mathbb{S}(s) \ge x,\ \mathbb{S}(t) \le y) \\
&= \frac{\partial}{\partial y}\phi\left(\frac{2x-y}{\sqrt{t}}\right)\frac{2}{\sqrt{t}} \\
&= \phi'\left(\frac{2x-y}{\sqrt{t}}\right)\frac{-1}{\sqrt{t}}\frac{2}{\sqrt{t}} \\
&= \frac{2x-y}{\sqrt{t}}\phi\left(\frac{2x-y}{\sqrt{t}}\right)\frac{2}{t} \\
&= \sqrt{\frac{2}{\pi t^3}}(2x-y)\exp\left(-\frac{(2x-y)^2}{2t}\right) \qquad \text{for } \ 0 \le y \le x
\end{aligned}$$

where we used $\phi'(x) = -x\phi(x)$ in the next to last equality.

2. **Bonus Problem 1.** Suppose that $\mathbb{S}$ is standard Brownian motion on $(C[0,\infty), \mathcal{C}_{[0,\infty)})$, and let its distribution be denoted by $P = P_0$. Let $\mathbb{S}_\mu(t) \equiv \mathbb{S}(t) + \mu t$ be Brownian motion with drift μ, and let P_μ denote the distribution of $\mathbb{S}_\mu$ on $(C[0,\infty), \mathcal{C}_{[0,\infty)})$. Set $Y(t) \equiv \exp(\mu\mathbb{S}(t) - \mu^2 t/2)$. For $t > 0$ let $P_{0,t}$ and $P_{\mu,t}$ denote the distributions P_0 and P_μ restricted to $\mathcal{A}_t \equiv \{\mathbb{S}(s) : s \le t\}$. Show that the Radon - Nikodym derivative $dP_{\mu,t}/dP_{0,t} = Y(t)$.

   Solution: First consider a fixed $0 < s \le t$ and $B \in \mathcal{B}$. Then

   $$P_\mu(\mathbb{S}_\mu(s) \in B) = P(N(\mu s, s) \in B) = \int_B \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{(x-\mu s)^2}{2s}\right) dx\,.$$

   On the other hand, since Y is a martingale,

   $$\begin{aligned}
   \int_{[\mathbb{S}(s)\in B]} Y(t)dP_0 &= \int_{[\mathbb{S}(s)\in B]} Y(s)dP_0 \\
   &= \int_B \exp(\mu x - \frac{1}{2}\mu^2 s)\frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{x^2}{2s}\right) dx \\
   &= \int_B \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{(x-\mu s)^2}{2s}\right) dx\,.
   \end{aligned}$$

   Thus

   $$P_\mu(\mathbb{S}_\mu(s) \in B) = \int_{[\mathbb{S}(s)\in B]} Y(t)dP_0 = \int_{[\mathbb{S}(s)\in B]} Y(s)dP_0\,.$$

   Now $\mathbb{S}_\mu(t) - \mathbb{S}_\mu(s) \sim N(\mu(t-s), t-s)$, so (a) continues to hold for $\mathbb{S}_\mu(t) - \mathbb{S}_\mu(s)$: i.e. for $B \in \mathcal{B}$,

   $$\begin{aligned}
   P_\mu(\mathbb{S}_\mu(t) - \mathbb{S}_\mu(s) \in B) &= \int_{[\mathbb{S}_\mu(t)-\mathbb{S}_\mu(s)\in B]} [Y(t)/Y(s)]dP_0 \\
   &= \int_{[\mathbb{S}_\mu(t)-\mathbb{S}_\mu(s)\in B]} \exp(\mu(\mathbb{S}(t)-\mathbb{S}(s)) - \mu^2(t-s))dP_0\,.
   \end{aligned}$$

   Now let $m \ge 1$, $t_{mi} = it/2^m$, $i = 0, 1, \ldots, 2^m$, and set

   $$X_i \equiv \mathbb{S}_\mu(t_{mi}) - \mathbb{S}_\mu(t_{m,i-1})\,,$$

and for $B_i \in \mathcal{B}$, $i = 1, \ldots, 2^m$, let

$$D_\mu \equiv \{\mathbb{S}_\mu(t_{mi}) - \mathbb{S}_\mu(t_{m,i-1}) \in B_i,\ i = 1, \ldots, 2^m\} = \cap_{i=1}^{2^m} \{\mathbb{S}_\mu(t_{mi}) - \mathbb{S}_\mu(t_{m,i-1}) \in B_i\}\,.$$

Now the X_i's are i.i.d. $N(\mu t/2^m, t/2^m)$, so that

$$\begin{aligned}
P_\mu(D_\mu) &= P_\mu(\mathbb{S}_\mu(t_{mi}) - \mathbb{S}_\mu(t_{m,i-1}) \in B_i,\ i = 1, \ldots, 2^m) \\
&= \prod_{i=1}^{2^m} P_\mu(X_i \in B_i) \\
&= \prod_{i=1}^{2^m} \int_{[X_i \in B_i]} \exp(\mu(\mathbb{S}(t_{mi}) - \mathbb{S}(t_{m,i-1})) - (1/2)\mu^2 t/2^m) dP_0 \qquad \text{by (b)} \\
&= \int_D \prod_{i=1}^{2^m} \exp(\mu(\mathbb{S}(t_{mi}) - \mathbb{S}(t_{m,i-1})) - (1/2)\mu^2 t/2^m) dP_0 \\
&\qquad \text{by independence of the } X_i' s \\
&= \int_D \exp(\mu \mathbb{S}(t) - (1/2)\mu^2 t) dP_0 \\
&= \int_D Y(t) dP_0\,.
\end{aligned}$$

But the sigma - field generated by all events of the form D for $B_i \in \mathcal{B}$ and $m \geq 1$ is just $\mathcal{A}_t$. Hence it follows from the equality in the last display that for $A \in \mathcal{A}_t$,

$$P_\mu(A) = \int_A Y(t) dP_0\,.$$

Thus $Y(t) = dP_{\mu,t}/dP_{0,t}$.

3. **Bonus Problem 2.** Suppose that $\mathbb{S}_\mu$ is Brownian motion with drift $\mu > 0$ as in problem 2, and let $\tau \equiv \inf\{t > 0 : \mathbb{S}_\mu(t) = a\}$, $a > 0$. Use the result of bonus problem 1 together with results from problem set 6 to find the distribution of τ. You should find that

$$\begin{aligned}
P_\mu(\tau > t) &= P_\mu(\mathbb{S}_\mu(s) < a,\ 0 \leq s \leq t) \\
&= \Phi\left(\frac{a - \mu t}{\sqrt{t}}\right) - e^{2\mu a} \Phi\left(\frac{-a - \mu t}{\sqrt{t}}\right)
\end{aligned}$$

and

$$f_\tau(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{(a-\mu t)^2}{2t}\right) \qquad \text{for } t \geq 0\,.$$

This is the *inverse Gaussian* density. [Note that this reduces to the density of τ_a from problem set 6 when $\mu = 0$.

Solution 1: By using the joint density found in the first problem in the fifth line,

$$\begin{aligned}
P_\mu(\tau > t) &= P_\mu(M_\mu(t) \equiv \sup_{0\leq s\leq t} \mathbb{S}_\mu(s) < a) \\
&= \int_{[M_\mu(t)<a]} dP_\mu \\
&= \int_{[M(t)<a]} Y(t) dP_0 \\
&= \int_{[M(t)<a]} \exp(\mu\mathbb{S}(t) - (1/2)\mu^2 t) dP_0 \\
&= \int_{x=0}^{a}\int_{y=-\infty}^{x} \exp(\mu y - (1/2)\mu^2 t) \\
&\qquad\qquad \cdot\sqrt{\frac{2}{\pi t^3}}(2x-y)\exp\left(-\frac{(2x-y)^2}{2t}\right) dydx \\
&= \Phi\left(\frac{a-\mu t}{\sqrt{t}}\right) - e^{2\mu a}\Phi\left(\frac{-a-\mu t}{\sqrt{t}}\right)\,,
\end{aligned}$$

and a differentiation to get the density completes the proof. The last equality here is checked as follows: first write

$$\begin{aligned}
&\int_{x=0}^{a}\int_{y=-\infty}^{x} \exp(\mu y - (1/2)\mu^2 t) \\
&\qquad\qquad \cdot\sqrt{\frac{2}{\pi t^3}}(2x-y)\exp\left(-\frac{(2x-y)^2}{2t}\right) dydx \\
\equiv\ & \int_{x=0}^{a}\int_{y=-\infty}^{x} g(x,y;t,\mu) dydx \\
=\ & \int_{x=0}^{a}\int_{y=-\infty}^{0} g(x,y;t,\mu) dydx
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_{x=0}^{a}\int_{y=0}^{x} g(x,y;t,\mu)dydx \\
&= \int_{y=0}^{a}\int_{x=y}^{a} g(x,y;t,\mu)dxdy \\
&\quad + \int_{y=-\infty}^{0}\int_{x=0}^{a} g(x,y;t,\mu)dxdy \\
&\equiv I + II\,.
\end{aligned}
$$

Now

$$
\begin{aligned}
I &= \int_{y=0}^{a}\int_{x=y}^{a} g(x,y;t,\mu)dxdy \\
&= \int_{y=0}^{a}\left(\int_{x=y}^{a}\frac{(2x-y)}{\sqrt{t}}\exp\left(-\frac{(2x-y)^2}{2t}\right)\frac{1}{\sqrt{t}}dx\right)\exp(\mu y-(1/2)\mu^2 t)\sqrt{\frac{2}{\pi t}}dy \\
&= \int_{y=0}^{a}\left(\exp\left(-\frac{y^2}{2t}\right)-\exp\left(-\frac{(2a-y)^2}{2t}\right)\right)\exp(\mu y-(1/2)\mu^2 t)\frac{1}{2}\sqrt{\frac{2}{\pi t}}dy \\
&= P(0\le N(\mu t,t)\le a)-e^{2\mu a}P(0\le N(2a+\mu t,t)\le a)\,.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
II &= \int_{y=-\infty}^{0}\int_{x=0}^{a} g(x,y;t,\mu)dxdy \\
&= \int_{y=-\infty}^{0}\left(\int_{x=0}^{a}\frac{(2x-y)}{\sqrt{t}}\exp\left(-\frac{(2x-y)^2}{2t}\right)\frac{1}{\sqrt{t}}dx\right)\exp(\mu y-(1/2)\mu^2 t)\sqrt{\frac{2}{\pi t}}dy \\
&= \int_{y=-\infty}^{0}\left(\exp\left(-\frac{y^2}{2t}\right)-\exp\left(-\frac{(2a-y)^2}{2t}\right)\right)\exp(\mu y-(1/2)\mu^2 t)\frac{1}{2}\sqrt{\frac{2}{\pi t}}dy \\
&= P(-\infty< N(\mu t,t)\le 0)-e^{2\mu a}P(-\infty< N(2a+\mu t,t)\le 0)\,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
I+II &= P(-\infty< N(\mu t,t)\le a)-e^{2\mu a}P(-\infty< N(2a+\mu t,t)\le a) \\
&= \Phi\left(\frac{a-\mu t}{\sqrt{t}}\right)-e^{2\mu a}\Phi\left(\frac{-a-\mu t}{\sqrt{t}}\right)\,.
\end{aligned}
$$

Solution 2: Here is a slightly simpler way using the martingale property of Y which involves almost no calculation

$$\begin{aligned}
P_\mu(\tau_\mu \le t) &= \int_{[\tau_\mu \le t]} dP_\mu = \int_{[\tau \le t]} Y(t)\,dP_0 \\
&= \int_{[\tau \le t]} Y(\tau)\,dP_0 \\
&\qquad \text{by optional sampling since } \tau \text{ is } \mathcal{A}_\tau - \text{measurable} \\
&= \int_{[\tau \le t]} \exp(\mu a - (1/2)\mu^2 \tau)\,dP_0 \qquad \text{since } \mathbb{S}_\tau) = a \\
&= \int_{[s \le t]} \exp(\mu a - (1/2)\mu^2 s)\frac{a}{\sqrt{2\pi s^3}} \exp\left(-\frac{a^2}{2s}\right) ds \\
&\qquad \text{by the computation of the density of } \tau \text{ under } P_0 \\
&\qquad \text{from problem set \#6} \\
&= \int_0^t \frac{a}{\sqrt{2\pi s^3}} \exp\left(-\frac{(a-\mu s)^2}{2s}\right) ds\,.
\end{aligned}$$

Thus under P_μ the stopping time $\tau = \tau_a$ has the *inverse Gaussian* density

$$f_\tau(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{(a-\mu t)^2}{2t}\right)\,, \quad t > 0\,.$$