Statistics 523, Problem Set 6 Solutions

Wellner; 5/14/99

1. Let $\tau_a \equiv \inf\{t > 0 : \mathbb{S}(t) = a > 0\}$ where $\mathbb{S}$ is standard Brownian motion. Define $\phi(s, a) \equiv E\exp(-s\tau_a)$. Use the strong Markov property of $\mathbb{S}$ to show that for $a, b > 0$ we have $\phi(s, a+b) = \phi(s, a)\phi(s, b)$. Use this to deduce via scaling properties of $\mathbb{S}$ that $\phi(s, a) = \exp(-c\sqrt{s}a)$ for some $c > 0$. [Note problem 37.10, Billingsley, page 550; or Remark 8.3, page 96, Karatzas and Shreeve (1991).]

   Solution: Let $\tau_a \equiv \inf\{t > 0 : \mathbb{S}(t) = a\}$ for $a > 0$. Then by the strong Markov property of BM

$$\widetilde{\mathbb{S}}(t) \equiv \mathbb{S}(\tau_a + t) - \mathbb{S}(\tau_a)$$

   is standard Brownian motion, and is independent of $\mathcal{A}_{\tau_a}$. Thus

$$\tilde{\tau}_b \equiv \inf\{t > 0 : \widetilde{\mathbb{S}}(t) = b\}$$

   is independent of $\mathcal{A}_{\tau_a}$, and we have

$$\begin{aligned} \phi(s, a+b) &= Ee^{-s\tau_{a+b}} \\ &= Ee^{-s[\tau_a + (\tau_{a+b} - \tau_a)]} \\ &= E\{e^{-s\tau_a}e^{-s(\tau_{a+b}-\tau_a)}\} \\ &= E\{e^{-s\tau_a}e^{-s\tilde{\tau}_b}\} \\ &= E\{e^{-s\tau_a}\}E\{e^{-s\tilde{\tau}_b}\} \\ &= \phi(s, a)\phi(s, b)\,. \end{aligned}$$

   Set $\psi_s(a) \equiv \phi(s, a)$; thus we have shown that

$$\psi_s(a+b) = \psi_s(a)\psi_s(b)\,. \tag{1}$$

   Furthermore, $\psi_s(a) \equiv \phi(s, a)$ is clearly monotone decreasing as a function of a, and the only solution of the functional equation (1) is of the form

$$\phi(s, a) = \psi_s(a) = \exp(d_s a)\,. \tag{2}$$

Now by Brownian scaling it follows that

$$\begin{aligned} \tau_a &= \inf\{t > 0 : \ \mathbb{S}(t) = a\} \\ &=_d \inf\{t > 0 : \ c^{-1}\mathbb{S}(c^2 t) = a\} \\ &= \inf\{\frac{s}{c^2} > 0 : \mathbb{S}(s) = ac\} \\ &= \frac{1}{c^2}\tau_{ca}\,. \end{aligned}$$

Thus by (2)

$$\phi(s, a) = Ee^{-s\tau_a} = Ee^{-\tau_{a\sqrt{s}}} = \exp(d_1 a\sqrt{s})\,.$$

Since ϕ is clearly a decreasing function of both s and a, it follows that $d_1 = -c$ for some $c > 0$.

2. (a) What is the relationship of $\sup_{0\le s\le t}\mathbb{S}(s)$ to τ_a in problem 1.
(b) Use this to find the exact distribution of τ_a, and then use this to check the computation of $\phi(s, a) = E\exp(-s\tau_a)$ of problem 1.
(c) Show that $E(\tau_a) = \infty$
(d) Show that $\tau_a =_d a^2/Z^2$ where $Z \sim N(0, 1)$.

Solution: (a) As we saw in class on 5/10-12,

$$P(\sup_{0\le s\le t}\mathbb{S}(s) > a) = P(\tau_a \le t) = 2P(Z \ge a/\sqrt{t}) \tag{3}$$

where the last equality holds by the reflection principle.
(b) By part (a),

$$\begin{aligned} P(\tau_a \le t) &= 2\int_{a/\sqrt{t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\ &= \int_0^t \frac{a}{\sqrt{2\pi y^3}} \exp(-\frac{a^2}{2y}) dy \\ &= \int_0^t f_a(y) dy \end{aligned}$$

by using the change of variables $z^2/2 = a^2/(2y)$, where the density function f_a of τ_a is given by

$$f_a(y) = \frac{a}{\sqrt{2\pi y^3}} \exp(-\frac{a^2}{2y}) 1_{(0,\infty)}(y)\,.$$

(c) It follows from (b) that

$$\begin{aligned}
\phi(s,a) &= E\exp(-s\tau_a) = \int_0^\infty \exp(-sy) f_a(y)dy \\
&= \int_0^\infty \exp(-sy)\frac{a}{\sqrt{2\pi y^3}}\exp(-\frac{a^2}{2y})dy \\
&= \exp(-a\sqrt{2s})\,,
\end{aligned}$$

from your favorite table of laplace transforms (see e.g. Abramowitz and Stegun (1965), *Handbook of Mathematical Functions*, page 1026, (29.3.82); or via *Mathematica* or *Maple*. Thus $c = \sqrt{2}$.

Another way to see this without the exact form of the density and the above computation is as follows: Consider $\tau^* \equiv \tau_1 \wedge \tau_{-1}$. Then $E\exp(-s\tau_1) = \exp(-c\sqrt{s}) = E\exp(-s\tau_{-1})$ by symmetry, and hence

$$\begin{aligned}
e^{-c\sqrt{s}} &= E\exp(-s\tau_1) \\
&= E\exp(-s\tau_1)1_{[\tau_1<\tau_{-1}]} + E\exp(-s\tau_1)1_{[\tau_1>\tau_{-1}]} \\
&= E\exp(-s\tau^*)1_{[\tau_1<\tau_{-1}]} + E\exp(-s\tau^*)1_{[\tau_1>\tau_{-1}]}\exp(-c\sqrt{s}2) \\
&\qquad \text{by Strong Markov} \\
&= b_s + b_s\exp(-c\sqrt{s}2) \\
&\qquad \text{by symmetry}\,.
\end{aligned}$$

Hence

$$b_s \equiv E\exp(-s\tau^*)1_{[\tau_1<\tau_{-1}]} = \frac{\exp(-c\sqrt{s})}{1+\exp(-c\sqrt{s}2)} = \frac{1}{\exp(c\sqrt{s})+\exp(-c\sqrt{s})}$$

and

$$\begin{aligned}
f(s) &\equiv E\exp(-s\tau^*) \\
&= E\exp(-s\tau^*)1_{[\tau_1<\tau_{-1}]} + E\exp(-s\tau^*)1_{[\tau_1>\tau_{-1}]} \\
&= 2b_s = \frac{1}{\cosh(c\sqrt{s})}
\end{aligned}$$

by (b) and symmetry. Now differentiate $f(s)$ in (c) to get $f'(0) = E\tau^*$:

$$f'(s) = \frac{-1}{[\cosh(c\sqrt{s})]^2}(-\sinh(c\sqrt{s}))c\frac{1}{2}s^{-1/2}$$

$$
\begin{aligned}
&= \frac{1}{[\cosh(c\sqrt{s})]^2}\frac{\sinh(c\sqrt{s})}{c\sqrt{s}}\frac{c^2}{2} \\
&\to 1\cdot 1\cdot \frac{c^2}{2} = \frac{c^2}{2} = E\tau^*\,.
\end{aligned}
$$

But $\{\mathbb{S}^2(0)-0, \mathbb{S}^2(\tau^*)-\tau^*\} = \{0, \mathbb{S}^2(\tau^*)-\tau^*\}$ is a two-term martingale, so that

$$\frac{c^2}{2} = E\tau^* = E\mathbb{S}^2(\tau^*) = 1\,.$$

Hence $c = \sqrt{2}$.

(c) From the form of the density function

$$
\begin{aligned}
E(\tau_a^r) &= \int_0^\infty y^r f_a(y)dy \\
&= \int_0^\infty y^r \frac{a}{\sqrt{2\pi y^3}}\exp(-\frac{a^2}{2y})dy \\
&= \int_0^\infty \frac{a}{\sqrt{2\pi}y^{3/2-r}}\exp(-\frac{a^2}{2y})dy \\
&\geq \int_M^\infty \frac{a}{\sqrt{2\pi}y^{3/2-r}}\exp(-\frac{a^2}{2y})dy \qquad \text{for every } M > 0 \\
&\geq \exp(-\frac{a^2}{2M})dy \int_M^\infty \frac{a}{\sqrt{2\pi}y^{3/2-r}}dy \\
&= \infty
\end{aligned}
$$

for every $r \geq 1/2$. Similarly, $E(\tau_a^r) < \infty$ for $0 < r < 1/2$.

(d) Note that if $Z \sim N(0,1)$, then

$$P(a^2/Z^2 \leq t) = P(Z^2 \geq a^2/t) = 2P(Z \geq a/\sqrt{t})$$

so that $\tau_a =_d a^2/Z^2$ in view of (3).

Remark: From

$$E\exp(-s\tau_a) = \exp(-\sqrt{2s}a)$$

we see that the characteristic function of τ_a is given by

$$E\exp(it\tau_a) = \exp(-\sqrt{-it}a) = \exp(-ai|t|^{1/2}\tan(\pi/4)\text{sign}(t));$$

this is the *completely assymmetric stable law* of index $\alpha = 1/2$.

3. PfS, Exercise 12.9.1, page 273. Show that $V_n(2) \to_{a.s.} 1$ if either (i) $\sum_{n=1}^{\infty} \|\mathcal{P}_n\| < \infty$ or (ii) the $\mathcal{P}_n$ are nested with $\|\mathcal{P}_n\| \to 0$.

   Solution: As suggested in the hint, we first compute $E[SS_n - 1]^2$:

$$\begin{aligned}
E[SS_n - 1]^2 &= E\left\{\sum_{k=1}^{n}\left[(\mathbb{S}(t_{n,k}) - \mathbb{S}(t_{n,k-1}))^2 - (t_{n,k} - t_{n,k-1})\right]\right\}^2 \\
&= E\left\{\sum_{k=1}^{n}\left[(\mathbb{S}(t_{n,k}) - \mathbb{S}(t_{n,k-1}))^2 - (t_{n,k} - t_{n,k-1})\right]\right. \\
&\qquad \left.\sum_{k'=1}^{n}\left[(\mathbb{S}(t_{n,k'}) - \mathbb{S}(t_{n,k'-1}))^2 - (t_{n,k'} - t_{n,k'-1})\right]\right\} \\
&= \sum_{k=1}^{n}\sum_{k'=1}^{n} E\left\{\left[(\mathbb{S}(t_{n,k}) - \mathbb{S}(t_{n,k-1}))^2 - (t_{n,k} - t_{n,k-1})\right]\right. \\
&\qquad \left.\left[(\mathbb{S}(t_{n,k'}) - \mathbb{S}(t_{n,k'-1}))^2 - (t_{n,k'} - t_{n,k'-1})\right]\right\} \\
&= \sum_{k=1}^{n} E\left\{\left[(\mathbb{S}(t_{n,k}) - \mathbb{S}(t_{n,k-1}))^2 - (t_{n,k} - t_{n,k-1})\right]^2\right\} \\
&= \sum_{k=1}^{n}(t_{n,k} - t_{n,k-1})^2 Var(Z^2) \qquad Z \sim N(0,1)\,, \\
&= 2\sum_{k=1}^{n}(t_{n,k} - t_{n,k-1})^2 \\
&\le 2\max_{1\le k\le n}(t_{n,k} - t_{n,k-1})\sum_{k=1}^{n}(t_{n,k} - t_{n,k-1}) \\
&= 2\|\mathcal{P}_n\| \cdot 1
\end{aligned}$$

   since $\sum_{k=1}^{n}(t_{n,k} - t_{n,k-1}) = 1$. Now by Markov's inequality it follows that

$$P(|SS_n - 1| > \epsilon) \le \frac{E[SS_n - 1]^2}{\epsilon^2} \le \frac{2\|\mathcal{P}_n\|}{\epsilon^2}$$

   and hence

$$\sum_{n=1}^{\infty} P(|SS_n - 1| > \epsilon) \le 2\epsilon^{-2}\sum_{n=1}^{\infty}\|\mathcal{P}_n\| < \infty$$

by the hypothesis in (i). Thus by Borel-Cantelli we conclude that

$$P(|SS_n - 1| > \epsilon \ \ i.o.) = 0\,;$$

i.e. $SS_n \to_{a.s.} 1$.

If the partitions are nested, then $\{SS_n\}_{n \geq 1}$ is a reverse martingale:

$$E(SS_{n-1}|SS_n, SS_{n+1}, \ldots) = SS_n \qquad \text{a.s.}\,;$$

i.e.

$$(4) \qquad E(SS_{n-1} - SS_n|SS_n, SS_{n+1}, \ldots) = 0 \qquad \text{a.s.}\,.$$

Once (4) is proved, that $SS_n \to_{a.s.} 1$ follows from the reverse-martingale convergence theorem: $SS_n \to_{a.s.}$something, and by theorem 18.3.2 "something" is exactly the limit in L_1 (or L_2). Since $SS_n \to_2 1$, we conclude that $SS_n \to_{a.s.} 1$.

To see that the reverse martingale property holds, note that since the partitions are nested, by inserting intermediate partitions if necessary, we may suppose that the number of partitioning points at the $n-$th state (including 0 and 1) is n. Moreover, we can list the points in the order in which they appear to give a new partition as $t_1, t_2, \ldots \in [0,1]$ so that the $n-$th partition has division points $t_1, \ldots, t_n$. (Note that these points are *not* ordered as $t_1 < \ldots < t_n$; in fact we can take $t_1 = 0, t_2 = 1, t_3, \ldots$ so that $t_1 = 0 < t_j < 1 = t_2$ for $j = 3, 4, \ldots$. In this scheme, t_{n+1} is the point added in going from $\mathcal{P}_n$ to $\mathcal{P}_{n+1}$. Now suppose that Y is a Rademacher random variable independent of $\mathbb{S}$: $P(Y = \pm 1) = 1/2$. Then

$$\tilde{\mathbb{S}}(s) \equiv \mathbb{S}(s \wedge t_n) + Y(\mathbb{S}(s) - \mathbb{S}(s \wedge t_n))$$

is again a standard Brownian motion process ($\tilde{\mathbb{S}}$ is the Brownian motion $\mathbb{S}$ run up to t_n, and then with the independent Brownian motion $\{Y(\mathbb{S}(s + t_n) - \mathbb{S}(t_n)) : \ s \geq 0\}$ pasted on beyond t_n), and $\tilde{\mathbb{S}}$ has the same quadratic variations $\widetilde{SS}_n$, $\widetilde{SS}_{n+1}, \ldots$ as $\mathbb{S}$, while $SS_{n-1} - SS_n$ is replaced by $Y(SS_{n-1} - SS_n)$: letting $t_- < t_n < t_+$ be the points of the partition at stage $n-1$ immediately to the right and left of t_n, it follows that

$$\begin{aligned} SS_{n-1} - SS_n &= [\mathbb{S}(t_+) - \mathbb{S}(t_-)]^2 - [\mathbb{S}(t_+) - \mathbb{S}(t_n)]^2 - [\mathbb{S}(t_n) - \mathbb{S}(t_-)]^2 \\ &= 2[\mathbb{S}(t_+) - \mathbb{S}(t_n)][\mathbb{S}(t_n) - \mathbb{S}(t_-)]\,, \end{aligned}$$

while, on the other hand,

$$\begin{aligned}\widetilde{SS}_{n-1}-\widetilde{SS}_n &= 2[\tilde{\mathbb{S}}(t_+)-\tilde{\mathbb{S}}(t_n)][\tilde{\mathbb{S}}(t_n)-\tilde{\mathbb{S}}(t_-)]\\ &= 2[Y(\mathbb{S}(t_+)-\mathbb{S}(t_n)][\mathbb{S}(t_n)-\mathbb{S}(t_-)]\\ &= Y(SS_{n-1}-SS_n)\,.\end{aligned}$$

Thus to prove (4), it suffices to show that

$$E(Y(SS_{n-1}-SS_n)|SS_n,SS_{n+1},\ldots)=0\,.$$

But by conditioning first on SS_{n-1} we compute

$$\begin{aligned}&E(Y(SS_{n-1}-SS_n)|SS_n,SS_{n+1},\ldots)\\ &\quad= E\left\{E(Y(SS_{n-1}-SS_n)|SS_{n-1},SS_n,SS_{n+1},\ldots)\Big|SS_n,SS_{n+1},\ldots\right\}\\ &\quad= E\left\{(SS_{n-1}-SS_n)\cdot E(Y|SS_{n-1},SS_n,SS_{n+1},\ldots)\Big|SS_n,SS_{n+1},\ldots\right\}\\ &\quad= E\left\{(SS_{n-1}-SS_n)\cdot 0\Big|SS_n,SS_{n+1},\ldots\right\}\\ &\quad= 0\,.\end{aligned}$$

Remark. It follows from this exercise that if $\mathbb{S}_\sigma$ is Brownian motion with variance σ^2 per unit time defined by $\mathbb{S}_\sigma(t)=\sigma\mathbb{S}(t)=_d\mathbb{S}(\sigma^2 t)$, $t\ge 0$, then for partitions $\mathcal{P}_n$ of $[0,t]$ the quadratic variation

$$V_n(2)(\mathbb{S}_\sigma)(t)\equiv\sum_{k=1}^n|\mathbb{S}_\sigma(t_{n,k})-\mathbb{S}_\sigma(t_{n,k-1})|^2$$

satisfies

$$V_n(2)(\mathbb{S}_\sigma)(t)\to_{a.s.}\sigma^2 t$$

if either $\sum_{n=1}^\infty\|\mathcal{P}_n\|<\infty$ or the partitions are nested. Thus if P_σ denotes the law of $\mathbb{S}_\sigma$ on $(C[0,\infty),\mathcal{C}_{[0,\infty)})$, $P_{\sigma'}\perp P_\sigma$ for $\sigma'\ne\sigma$.