Statistics 523, Problem Set 5 Solutions

Wellner; 5/6/99

1. PfS, Exercise 12.3.1, page 255. In proving (17) could you get away with less than the LIL for Brownian motion at infinity?

   Solution:

   (11) If $\mathbb{V}$ is a Brownian bridge process and $Z \sim N(0,1)$ is independent of $\mathbb{V}$, then the process S defined by $S(t) = \mathbb{V}(t) + tZ$, $0 \le t \le 1$ is a Brownian motion:
   We did this one in class, but here it is again: The process S is clearly Gaussian because $\mathbb{V}$ and Z are. Also $E(S(t)) = E(\mathbb{V}(t)) + tE(Z) = 0 + t0 = 0$, and

   $$\begin{aligned} Cov(S(s), S(t)) &= Cov(\mathbb{V}(s), \mathbb{V}(t)) + stE(Z^2) \\ &= s \wedge t - st + st \\ &= s \wedge t \, . \end{aligned}$$

   Hence S is Brownian motion (on $[0,1]$).

   (12) The process $Z(t) = S(at)/\sqrt{a}$ is clearly Gaussian with mean zero because S is. Furthermore,

   $$Cov(Z(s), Z(t)) = \frac{1}{a} Cov[S(as), S(at)] = \frac{1}{a} Cov[S(as), S(at)] = \frac{1}{a}(as) \wedge (at) = s \wedge t \, .$$

   Thus Z is standard Brownian motion

   (13) If S is standard Brownian motion, then the process $B(t) = S(a+t) - S(a)$, $t \ge 0$ is Gaussian with mean zero, and, if $0 \le s \le t < \infty$, then

   $$\begin{aligned} Cov(B(s), B(t)) &= Cov(S(a+s), S(a+t)) - Cov(S(a+s), S(a)) \\ &\qquad - Cov(S(a+t), S(a)) + Cov(S(a), S(a)) \\ &= (a+s) \wedge (a+t) - (a+s) \wedge a - (a+t) \wedge a + a \\ &= (a+s) - a - a + a \\ &= s = s \wedge t \, . \end{aligned}$$

Thus B is standard Brownian motion.

(14) If $\mathbb{U}^{(1)}$ and $\mathbb{U}^{(2)}$ are two independent Brownian bridge processes and $a \in [0,1]$, then $\mathbb{A}(t) = \sqrt{1-a}\mathbb{U}^{(1)}(t) \pm \sqrt{a}\mathbb{U}^{(2)}(t)$ is clearly Gaussian with mean zero. Moreover

$$\begin{aligned} Cov(\mathbb{A}(s), \mathbb{A}(t)) &= (1-a)Cov(\mathbb{U}^{(1)}(s), \mathbb{U}^{(1)}(t)) + aCov(\mathbb{U}^{(2)}(s), \mathbb{U}^{(2)}(t)) \\ &= (1-a)(s \wedge t - st) + a(s \wedge t - st) \\ &= (s \wedge t - st)\,. \end{aligned}$$

Thus $\mathbb{A}$ is a Brownian bridge process on $[0,1]$.

(15) If $\mathbb{U}$ is a Brownian bridge process on $[0,1]$, then the process $Z(t) = [\mathbb{U}(t) + \mathbb{U}(1-t)]\sqrt{2}$, $0 \le t \le 1/2$ is a Brownian motion process since it is clearly mean 0 Gaussian, and, moreover, for $s \le t \le 1/2$,

$$\begin{aligned} Cov(Z(s), Z(t)) &= \frac{1}{2}\{Cov[\mathbb{U}(s), \mathbb{U}(t)] + +Cov[\mathbb{U}(s), \mathbb{U}(1-t)] \\ &\quad + \; Cov[\mathbb{U}(t), \mathbb{U}(1-s)] + Cov[\mathbb{U}(1-s), \mathbb{U}(1-t)]\} \\ &= \frac{1}{2}\{s \wedge t - st + s - s(1-t) \\ &\quad + \; t - t(1-s) + (1-t) - (1-s)(1-t)\} \\ &= \frac{1}{2}\{s - st + s - s + st + t - t + st + s - st\} \\ &= \frac{1}{2}\{2s\} = s = s \wedge t\,. \end{aligned}$$

Thus Z is standard Brownian motion.

(16) If S is standard Brownian motion, then $\mathbb{U}(t) = (1-t)S(t/(1-t))$ is clearly mean 0 Gaussian; moreover, for $s \le t$,

$$\begin{aligned} Cov(\mathbb{U}(s), \mathbb{U}(t)) &= (1-s)(1-t)Cov(S(s/(1-s)), S(t/(1-t)) \\ &= (1-s)(1-t)s/(1-s) = s(1-t) \\ &= s \wedge t - st\,, \end{aligned}$$

and hence $\mathbb{U}$ is a Brownian bridge process. Note that $\mathbb{U}(t) \rightarrow_{a.s.} 0$ as $t \rightarrow 1$ since

$$\mathbb{U}(t) = (1-t)S(t/(1-t)) = t\frac{S(t/(1-t))}{t/(1-t)} \rightarrow_{a.s.} 1 \cdot 0 = 0$$

at $t \to 1$; this follows from the "law of large numbers" for S at infinity: $S(y)/y \to_{a.s.} 0$ as $y \to \infty$.

(17) If S is standard Brownian motion, then $B(t) \equiv tS(1/t)$ is also a standard Brownian motion. This follows because B is clearly a 0-mean Gaussian process with mean 0, and, for $s \le t$,

$$\begin{aligned} Cov[B(s), B(t)] &= stCov[S(1/s), S(1/t)] = st\,\{(1/s)] \wedge (1/t)\} \\ &= st(1/t) = s = s \wedge t\,, \end{aligned}$$

and hence B is standard Brownian motion. Also, by the strong law of large numbers for S, $S(y)/y \to_{a.s.} 0$ as $y \to \infty$, and therefore

$$B(t) = tS(1/t) = \frac{S(1/t)}{1/t} \to_{a.s.} 0$$

as $t \to 0$. Note that in both (16) and (17) we just used the "SLLN for Brownian motion" $S(y)/y \to_{a.s.} 0$ as $y \to \infty$ rather than the LIL

$$\limsup_{t\to\infty} \frac{|S(t)|}{\sqrt{2t \log\log t}} = 1 \qquad \text{a.s.}\,.$$

Of course the SLLN follows from the LIL, since

$$\frac{|S(t)|}{t} = \frac{\sqrt{2t\log\log t}}{t} \frac{|S(t)|}{\sqrt{2t\log\log t}}$$

where the first term converges to 0 and the second term has a finite limsup on t almost surely. But the SLLN is easier to prove than the LIL: since $S(n) = \sum_{j=1}^{n}[S(j) - S(j-1)]$ where the increments $S(j) - S(j-1)$ are i.i.d. $N(0,1)$, it follows from the SLLN that $S(n)/n \to_{a.s.} 0$. Therefore

$$\begin{aligned} \frac{S(y)}{y} &= \frac{S(y) - S([y]) + S([y])}{y} \\ &= \frac{S(y) - S([y])}{y} + \frac{S([y])}{[y]}\frac{[y]}{y} \\ &\to_{a.s.} 0 + 0 \cdot 1 = 0 \end{aligned}$$

by an easy argument since the first term in absolute value is stochastically dominated by $|Z|/y$ where Z is $N(0,1)$.

2. PfS, Exercise 12.3.4, page 255: Let $Z_0, Z_1, Z_2, \ldots$ be iid $N(0,1)$. Let $f_j(t) = \sqrt{2}\sin(j\pi t)$, for $j \geq 1$. Then

$$\mathbb{U}(t) \equiv \sum_{j=1}^{\infty} Z_j \frac{f_j(t)}{j\pi}, \qquad 0 \leq t \leq 1, \tag{1}$$

is a Brownian bridge.

Solution: First note that the collections of functions $\{f_j\}_{j\geq 1} \equiv \{\sqrt{2}\sin(\pi jt)\}_{j\geq 1}$ and $\{g_j\}_{j\geq 1} \equiv \{\sqrt{2}\cos(\pi jt)\}_{j\geq 1}$ are both orthonormal families in $L_2(0,1) = L_2([0,1],\lambda)$ where λ is Lebesgue measure: e.g., for $j = 1, 2, \ldots$,

$$\int_0^1 g_j^2(t)dt = 2\int_0^1 \cos^2(\pi jt)dt = \int_0^1 [1+\cos(2\pi jt)]dt = 1+\int_0^1 \cos(2\pi jt)dt = 1$$

while, for $j \neq k$,

$$\begin{aligned}
\int_0^1 g_j(t)g_k(t)dt &= 2\int_0^1 \cos(\pi jt)\cos(\pi kt)dt \\
&= \int_0^1 \{\cos(\pi t(j+k)) + \cos(\pi t(j-k))\}\,dt \\
&= 0 + 0 = 0\,.
\end{aligned}$$

In fact, both these families, together with the constant function 1, form an orthonormal basis for $L_2(0,1)$. Also note that

$$\int_0^t g_j(s)ds = \int_0^1 1_{[0,t]}(x)g_j(x)dx = \sqrt{2}\frac{\sin(\pi jt)}{j\pi}\,.$$

Hence it follows from (1), Fubini, independence of the Z_j's, $E(Z_j) = 0$, and Parseval's formula that

$$\begin{aligned}
E[\mathbb{U}(s)\mathbb{U}(t)] &= E\left\{\left(\sum_{j=1}^{\infty} Z_j \frac{f_j(s)}{j\pi}\right)\left(\sum_{j'=1}^{\infty} Z_{j'}\frac{f_{j'}(t)}{j'\pi}\right)\right\} \\
&= \sum_{j=1}^{\infty} \frac{f_j(s)f_j(t)}{j^2\pi^2} \\
&= \sum_{j=1}^{\infty} \int_0^1 1_{[0,s]}(x)g_j(x)dx \int_0^1 1_{[0,t]}(x)g_j(x)dx
\end{aligned}$$

$$\begin{aligned} &= \sum_{j=1}^{\infty} \int_0^1 1_{[0,s]}(x) g_j(x)dx \int_0^1 1_{[0,t]}(x) g_j(x)dx + st - st \\ &= \int_0^1 1_{[0,s]}(x) 1_{[0,t]}(x)dx - st \\ &= s \wedge t - st\,. \end{aligned}$$

Since $\mathbb{U}$ is clearly a mean 0 Gaussian process, $\mathbb{U}$ is a Brownian bridge process on $[0,1]$. By (11) of the preceding problem $S(t) \equiv \mathbb{U}(t) + tZ_0$ is Brownian motion on $[0,1]$. To see that

$$W^2 \equiv \int_0^1 \mathbb{U}^2(t)dt = \sum_{j=1}^{\infty} \frac{Z_j^2}{j^2\pi^2}\,,$$

note that by orthonormality of the family of functions $\{f_j\}_{j\ge 1}$ ($\int_0^1 f_j^2(t)dt = 1$, $\int_0^1 f_j^{(}t) f_k(t)dt = 0$ for $j \ne k$), it follows that

$$\begin{aligned} W^2 &= \int_0^1 \mathbb{U}^2(t)dt \\ &= \int_0^1 \sum_{j=1}^{\infty} Z_j \frac{f_j(t)}{j\pi} \sum_{j'=1}^{\infty} Z_{j'} \frac{f_{j'}(t)}{j'\pi}\, dt \\ &= \sum_{j=1}^{\infty}\sum_{j'=1}^{\infty} \frac{Z_j Z_{j'}}{jj'\pi^2} \int_0^1 f_j(t) f_{j'}(t)dt \\ &= \sum_{j=1}^{\infty} \frac{Z_j^2}{j^2\pi^2}\,. \end{aligned}$$

For a slightly different approach to this and other "goodness of fit" statistics, see Shorack and Wellner (1986), *Empirical Processes with Applications to Statistics*, Chapter 5, and especially section 5.3, pages 213-224.

3. PfS, Exercise 12.7.3, page 267: (a) Show that $E\tau^r \le 4r\Gamma(r)ab(a+b)^{2r-2}$ holds for $r = 2$.
(b) Show that $E\tau^r \le 4r\Gamma(r)ab(a+b)^{2r-2}$ holds for integral $r \ge 1$.
(c) Show that $E\tau^2 = a^2$ when $b = a$.

Solution: (a) The process $V_\theta(t) = \exp(\theta\mathbb{S}(t) - \theta^2 t/2)$ is a martingale for each fixed θ. Taking derivatives w.r.t. θ and evaluating them at $\theta = 0$ yields

$$\frac{d}{d\theta}V_\theta(t)|_{\theta=0} = V_\theta(t)(\mathbb{S}(t) - \theta t)|_{\theta=0} = \mathbb{S}(t)\,,$$

$$\frac{d^2}{d\theta^2}V_\theta(t)|_{\theta=0} = \left\{V_\theta(t)(\mathbb{S}(t) - \theta t)^2 - V_\theta(t)t\right\}|_{\theta=0} = \mathbb{S}^2(t) - t\,,$$

$$\begin{aligned}\frac{d^3}{d\theta^3}V_\theta(t)|_{\theta=0} &= \left\{V_\theta(t)(\mathbb{S}(t) - \theta t)^3 + V_\theta(t)2(\mathbb{S}(t) - \theta t)(-t) - V_\theta(t)t(\mathbb{S}(t) - \theta t)\right\}|_{\theta=0} \\ &= \mathbb{S}^3(t) - 3t\mathbb{S}(t)\,,\end{aligned}$$

and

$$\begin{aligned}\frac{d^4}{d\theta^4}V_\theta(t)|_{\theta=0} &= \left\{V_\theta(t)(\mathbb{S}(t) - \theta t)^4 + V_\theta(t)3(\mathbb{S}(t) - \theta t)^2(-t)\right. \\ &\quad -2tV_\theta(t)(\mathbb{S}(t) - \theta t)^2 + V_\theta(t)(2t^2) \\ &\quad \left.-tV_\theta(t)(\mathbb{S}(t) - \theta t)^2 + t^2V_\theta(t)\right\}|_{\theta=0} \\ &= \mathbb{S}^4(t) - 6t\mathbb{S}^2(t) + 3t^2\,.\end{aligned}$$

Hence by the optional sampling theorem and then the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}0 &= E\mathbb{S}^4(\tau) - 6E[\tau\mathbb{S}^2(\tau)] + 3E[\tau^2] \\ &\geq E\mathbb{S}^4(\tau) - 6\sqrt{E[\tau^2]}\sqrt{E[\mathbb{S}^4(\tau)]} + 3E[\tau^2] \\ &\equiv c^2 - 6cx + 3x^2\,.\end{aligned}$$

Now $3x^2 - 6cx + c^2$ has roots

$$x = \frac{6c \pm \sqrt{36c^2 - 12c^2}}{6} = c\left(1 + \frac{\sqrt{6}}{3}\right) \leq 2c\,.$$

Furthermore $3x^2 - 6cx + c^2 \leq 0$ on the interval between the roots, or

$$x^2 \leq 2cx - c^2/3 \leq 2cx \leq 2cc(1 + \sqrt{6}/3) \leq 4c^2\,,$$

and this yields

$$\begin{aligned} E(\tau^2) &\leq 4E\mathbb{S}^4(\tau) \\ &= 4\left\{ b^4\frac{a}{a+b} + a^4\frac{b}{a+b} \right\} \\ &= \frac{4ab}{a+b}\left(a^3+b^3\right) \\ &= 4ab(a^2-ab+b^2) \leq 8ab(a+b)^2 \,. \end{aligned}$$

In fact the density of $\tau = \tau_{ab}$ is given by

$$f_\tau(t) = \frac{2\pi}{(a+b)^2}\sum_{k=0}^{\infty}(-1)^k(2k+1)\cos(\frac{2k+1}{2}\pi\frac{|a-b|}{a+b})\exp\left(-\frac{(2k+1)^2}{2}\left(\frac{\pi}{a+b}\right)^2 t\right)\;;$$

this can be obtained by differentiating the expression given for the tail probabilities $P(\tau_{ab} > t)$ given in Feller (1971), page 342:

$$P(\tau_{ab} > t) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\exp\left(-\frac{(2k+1)^2}{2}\left(\frac{\pi}{a+b}\right)^2 t\right)\sin(\frac{2k+1}{2}\pi\frac{|a-b|}{a+b})\,.$$

As we already know from the martingale arguments, this yields

$$E\tau = \int_0^\infty t f_\tau(t)dt = \frac{8(a+b)^2}{\pi^3}\sum_{k=0}^{\infty}(-1)^k(2k+1)^{-3}\cos\left(\frac{2k+1}{2}\pi\frac{|a-b|}{a+b}\right) = ab\,,$$

and, similarly (but now new!),

$$\begin{aligned} E\tau^2 &= \int_0^\infty t^2 f_\tau(t)dt = \frac{32(a+b)^4}{\pi^5}\sum_{k=0}^{\infty}(-1)^k(2k+1)^{-5}\cos\left(\frac{2k+1}{2}\pi\frac{|a-b|}{a+b}\right) \\ &= \frac{(a+b)^4}{48}\left(\frac{|b-a|^2}{(b+a)^2}-1\right)\left(\frac{|b-a|^2}{(b+a)^2}+5\right) \\ &= \frac{1}{3}ab(a^2+3ab+b^2)\,, \end{aligned}$$

where we used

$$1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\cdots = \frac{5\pi^5}{1536}$$

and this result for $E(\tau^2)$ yields

$$Var(\tau) = \frac{1}{3}ab(a^2+b^2)\,.$$

Note that this agrees with the result from the martingale calculation in the case $b = a$: in this case $|\mathbb{S}(\tau)| = a$ with probability one, and it follows that

$$\begin{aligned} 0 &= E\mathbb{S}^4(\tau) - 6E[\tau\mathbb{S}^2(\tau)] + 3E[\tau^2] \\ &= a^4 - 6E[\tau a^2] + 3E[\tau^2] \\ &= a^4 - 6a^4 + 3E[\tau^2] \end{aligned}$$

so that $E[\tau^2] = (5/3)a^4$, and $Var(\tau) = (2/3)a^4$.

Notes: As several of you noted, the current problem statement in PfS contains several inaccuracies; in particular the definition of V_θ should be changed as used above, and the last part involving the case $b = a$ is incorrect. A proof of $E\tau^r \leq 4r\Gamma(r)ab(a+b)^{2r-2}$ holds for and $r \geq 1$ is contained in the Appendix of the following paper

> Rosenkrantz, W. (1967). On rates of convergence for the invariance principle. *Trans. Amer. Math. Soc.* **129**, 542-552.

His proof uses the form of the density given above.