Statistics 523, Problem Set 4 Solutions

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1. PfS, Exercise 15.1.5, page 340. Verify example 1.3 (Poisson convergence):
   (i) The Poisson(λ) characteristic function ϕ has

$$Log\phi(t) = \lambda(e^{it} - 1) = it\lambda + \lambda(e^{it} - 1 - it)\,.$$

   Thus $\beta = \lambda$ and $K = \lambda 1_{[1,\infty)}$ in (15.1.18).
   (ii) (Convergence). Suppose that the triangular array with $X_{nk} \sim (\mu_{nk}, \sigma^2_{nk})$ satisfies $\max_{k \le n} \sigma^2_{nk} \to 0$ and $\sum_1^n \sigma^2_{nk} \to \lambda \in (0,\infty)$. Then $S_n \equiv \sum_1^n X_{nk} \to_d \text{Poisson}(\lambda)$ if and only if

$$\sum_1^n \mu_{nk} \to \lambda \qquad \text{and} \qquad \sum_1^n \int_{[|x-1| \ge \epsilon]} x^2 dF_{nk}(x + \mu_{nk}) \to 0\,.$$

   Solution: (i) This part is easy: if $X \sim \text{Poisson}(\lambda)$, then

$$\begin{aligned} \phi(t) &= Ee^{itX} = \sum_{k=0}^{\infty} e^{itk} e^{-\lambda} \frac{\lambda^k}{k!} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{[\lambda e^{it}]^k}{k!} = e^{-\lambda} e^{\lambda e^{it}} \\ &= \exp(\lambda(e^{it} - 1)) = \exp(\lambda(e^{it} - 1 - it) + it\lambda)\,. \end{aligned}$$

   This is of the form of (15.1.18) with $\beta = \lambda$ and $K(x) = \lambda 1_{[1,\infty)}(x)$.
   (ii) By theorem 15.1.2, convergence of $S_n \equiv \sum_1^n X_{nk}$ to the Poisson limit in (i) occurs if and only if both

$$\mu_n \equiv \sum_1^n \mu_{nk} \to \lambda \qquad \text{and} \qquad K_n(x) = \sum_1^n \int_{-\infty}^x y^2 dF_{nk}(y + \mu_{nk}) \to_{sd} \lambda 1_{[1,\infty)}(x)\,.$$

But since we know that $\sum_1^n \sigma_{nk}^2 \to \lambda$, the second condition involving convergence of K_n is equivalent to

$$\sum_{k=1}^{n} \int_{[|x-1| \geq \epsilon]} y^2 dF_{nk}(y + \mu_{nk}) \to 0$$

for every $\epsilon > 0$.

2. PfS, Exercise 15.1.7, page 340. If ϕ is a characteristic function, the $\exp(c(\phi - 1))$ is an infinitely divisible characteristic function for all $c > 0$.

   Solution: Let $W_1, W_2, \ldots$ be i.i.d. with characteristic function $\phi = \phi_W$ (and corresponding distribution function F_W); then let $N(c)$ ~Poisson(c) be a Poisson random variable which is independent of $W_1, W_2, \ldots$. Then $S \equiv W_1 + \cdots + W_{N(c)}$ has characteristic function

$$\begin{aligned}
\phi_S(t) &= Ee^{itS} = E[E\{e^{itS}|N(c)\}] \\
&= E[\phi_W(t)^{N(c)}] = \sum_{k=0}^{\infty} \phi_W(t)^k \exp(-c)\frac{c^k}{k!} \\
&= \exp(-c) \sum_{k=0}^{\infty} \frac{c^k \phi_W(t)^k}{k!} \\
&= \exp(-c)\exp(c\phi_W(t)) = \exp(c(\phi_W(t) - 1)) \,.
\end{aligned}$$

   This distribution is clearly infinitely divisible: Let Y_{ni} be i.i.d. with characteristic function $\exp((c/n)(\phi_W(t) - 1))$. Then $Y_{n1} + \cdots + Y_{nn}$ has characteristic function $\exp(c(\phi_W(t) - 1))$.

3. PfS, Exercise 15.4.2, page 347:
   (a) State necessary and sufficient conditions on F for $F \in \mathcal{D}$(Cauchy).
   (b) Do the same for $F \in \mathcal{D}_N$(Cauchy).
   (c) Show by example that $\mathcal{D}_N$(Cauchy) is a proper subset of $\mathcal{D}$(Cauchy).

   (d) Note that a symmetric df F is in $\mathcal{D}$(Cauchy) if an only if $xP(X > x) = x(1 - F(x))$ is slowly varying.

   Solution: (a) First note that for the Cauchy$(0, 1)$ distribution we have $\phi(t) = \exp(-|t|)$. Hence by Theorem 15.3.1, $\theta = 0$ and $p = 1/2$. Thus by part (a) of theorem 15.4.1, $F \in \mathcal{D}(\text{Cauchy})$ if and only if both $U(x) \equiv E\{X^2 1_{[|X| \le x]}\} \in \mathcal{U}_1$ and $P(X > x)/P(|X| > x) \to 1/2$. By exercise 4.1, the condition involving U is equivalent to $P(|X| > x) \in \mathcal{U}_{-1}$, and this is, in turn, equivalent to

$$\frac{x^2 P(|X| > x)}{U(x)} \to 1\,.$$

(b) By Part (b) of theorem 15.4.1, $F \in \mathcal{D}_N(\text{Cauchy})$ if and only $xP(X > x) \to c/2$ and $xP(X < -x) \to c/2$ for some $c > 0$.
(c) Let

$$f(x) = \frac{\log |x|}{2x^2} 1_{[1,\infty)}(|x|)\,.$$

Then f is symmetric about 0, and, for $x \ge 1$,

$$\begin{aligned}
1 - F(x) &= P(X > x) = \frac{1}{2}\int_x^\infty \frac{\log y}{y^2}\,dy \\
&= \frac{1}{2}\int_{\log x}^\infty v e^{-v}\,dv \\
&\qquad \text{by the change of variables} \quad v = \log y \\
&= \frac{1}{2} P(\text{Gamma}(2,1) > \log x) \\
&= \frac{1}{2} P(\text{Poisson}(\log x) \le 1) \\
&= \frac{1}{2}\left(e^{-\log x} + \log x e^{-\log x}\right) \\
&= \frac{1}{2x}\left(1 + \log x\right)\,.
\end{aligned}$$

Note that this equals $1/2$ at $x = 1$, so that f is a density. Thus $P(|X| > x) = x^{-1}(1 + \log x)$ and hence both $P(|X| > x) \in \mathcal{U}_{-1}$ and $P(X > x)/P(|X| > x) = 1/2$ for all $x \in [0,\infty)$. Thus by (a), $F \in \mathcal{D}(\text{Cauchy})$. On the other hand, $xP(X > x) = xP(X < -x) = (1 + \log x)/2 \not\to c$, so $F \notin \mathcal{D}_N(\text{Cauchy})$. Also note that, for $x \ge 1$,

$$U(x) = \frac{1}{2}\int_{-x}^x y^2 f(y)\,dy = \int_1^x \log y\,dy = x(\log x - 1) + 1$$

satisfies $U \in \mathcal{U}_1$ while

$$\frac{x^2 P(|X| > x)}{U(x)} = \frac{x(1 + \log x)}{x(\log x - 1) + 1} \to 1\,.$$

(d) This part is easy: $P(X > x)/P(|X| > x) \to 1/2$ holds trivially for a symmetric df since $P(X > x)/P(|X| > x) = 1/2$ for all x, while the condition $P(|X| > x) \in \mathcal{U}_{-1}$ reads $2P(X > x) \in \mathcal{U}_{-1}$, which is easily equivalent to $P(X > x) \in \mathcal{U}_{-1}$.