Statistics 523, Final Exam Solutions

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1. (32 points). **Define** *four* of the following five terms:
(a) An *infinitely divisible* distribution (or random variable).
(b) A *stable distribution* (or random variable).
(c) A *stopping time* relative to a filtration $\{\mathcal{A}_t : 0 \le t < \infty\}$.
(d) The *strong Markov property* of a process $\{X(t) : 0 \le t < \infty\}$.
(e) A *tight* sequence of distribution functions $\{F_n\}$
or probability measures $\{P_n\}$.

   Solution: See PfS.

2. (36 points). Give careful **statements** of *three* of the following five theorems or results:
(a) The Berry-Esseen theorem.
(b) Donsker's theorem for the partial sum process $\{\mathbb{S}_n(t) :\ 0 \le t \le 1\}$.
(c) Donsker's theorem for the uniform empirical process
$\{\mathbb{U}_n(t) :\ 0 \le t \le 1\}$.
(d) Four properties of Brownian motion $\mathbb{S}$ on $[0, \infty)$.
(e) The Cramér - Lévy continuity theorem for characteristic functions.

   Solution: See PfS.

3. (48 points). A. Prove the following:
If $\mathbb{S}$ is standard Brownian motion on $[0, \infty)$, then

$$P(\sup_{0 \le s \le t} \mathbb{S}(s) \ge x) = 2P(\mathbb{S}(t) \ge x) = 2(1 - \Phi(x/\sqrt{t}))\,.$$

   B. For $a > 0$, $b \in R$, let

$$\tau \equiv \inf\{t > 0 : \mathbb{S}(t) = a + bt\}\,.$$

   Use the martingale $Y(t) = \exp(\theta\mathbb{S}(t) - \theta^2 t/2)$ to show that

$$E\exp(-\lambda\tau) = \exp(-a\{b + (b^2 + 2\lambda)^{1/2}\})\,.$$

[Hint: choose $\theta = b + (b^2 + 2\lambda)^{1/2}$.]

Solution: A. This was proved in Section 12.7, PfS, page 265.
B. Since the process Y is a mean 1 martingale, *if* application of the optional sampling theorem can be justified, then we have

$$\begin{aligned} 1 &= EY(\tau) = E\exp(\theta\mathbb{S}(\tau) - \theta^2\tau/2) \\ &= E\exp(\theta(a + b\tau) - \theta^2\tau/2) \\ &= E\exp(\theta a + (\theta b - \theta^2/2)\tau)\,, \end{aligned}$$

and hence

$$E\exp((\theta b - \theta^2/2)\tau) = \exp(-\theta a)\,,$$

and choosing $\theta = b + (b^2 + 2\lambda)^{1/2}$ as suggested in the hint, this yields

$$E\exp(-\lambda\tau) = \exp(-a(b + (b^2 + 2\lambda)^{1/2}))\,. \tag{1}$$

The potential difficulty is in verifying that the optional sampling theorem can be applied, and the possibility is the case $b > 0$, since then τ is even an extended-valued random variable. Note that the limit of the right side of (1) as $\lambda \to 0$ is

$$\exp(-a(b + |b|)) = \begin{cases} \exp(-a \cdot 0) = 1 & \text{if } b \le 0 \\ \exp(-2ab) < 1 & \text{if } b > 0 \end{cases}\,.$$

This agrees with our calculation in Theorem 12.7.2: if $b \ge 0$ and $a > 0$ then

$$P(\mathbb{S}(t) > bt + a \ \text{ for some } t \ge 0) = P(\tau < \infty) = \exp(-2ab)\,,$$

so that in the case $b > 0$ we have $P(\tau = \infty) = 1 - \exp(-2ab) > 0$. [Also note that the Laplace transform of an extended valued random variable τ is well-defined and the limit as $\lambda \to 0$ yields $P(\tau < \infty)$.]

Also, note that the limit of the right side of (1) as $b \to 0$ is $\exp(-a\sqrt{2\lambda})$ in agreement with the solution to problem 2, problem set 6.

What we need to apply the optional sampling theorem are: (a) $EY(\tau) < \infty$, and (b)

$$\liminf_{s\to\infty} E\{1_{[\tau > s]}Y(s)\} = 0\,.$$

Let K be fixed Now the stopping time $\tau_K \equiv \tau \wedge K$ is bounded by K and the elementary optional sampling theorem yields $1 = EY(\tau_K)$.

4. (42 points). Suppose that $X_1, \ldots, X_n$ are iid F_0 (continuous) and you form the statistic

$$T_n = \int_{-\infty}^{\infty} \sqrt{n}|\mathbb{F}_n(x) - F_0(x)|dF_0(x)\,.$$

(i) What is the limiting distribution of T_n in terms of a Brownian bridge process $\mathbb{U}$?
(ii) Is the limit the same if you replace T_n by

$$T_n = \int_{-\infty}^{\infty} \sqrt{n}|\mathbb{F}_n(x) - F_0(x)|d\mathbb{F}_n(x)\,.$$

Why or why not?

Solution: First note that by the basic inverse transformation

$$T_n =_d \int_{-\infty}^{\infty} |\mathbb{U}_n(F_0(x))|dF_0(x) = \int_0^1 |\mathbb{U}_n(t)|dt \equiv g(\mathbb{U}_n)\,,$$

by the change of variables $F_0(x) = t$. Then, defining

$$T \equiv \int_0^1 |\mathbb{U}(t)|dt = g(\mathbb{U})$$

for a Brownian bridge process $\mathbb{U}$, since the function $g : D[0,1] \to R$ is continuous with respect to the supremum metric $\|\cdot\|_\infty$,

$$T_n = g(\mathbb{U}_n) \to_d g(\mathbb{U}) = T \tag{2}$$

follows from Donsker's theorem for $\mathbb{U}_n$. Even more intuitively, if $\mathbb{U}_n$ and $\mathbb{U}$ have been constructed on a common probability space so that $\|\mathbb{U}_n - \mathbb{U}\|_\infty \to_p 0$, then

$$|T_n - T| \;\le\; \int_0^1 |\mathbb{U}_n(t) - \mathbb{U}(t)|\,dt \le \|\mathbb{U}_n - \mathbb{U}\|_\infty \to_p 0\,,$$

and hence (2) continues to hold for any version of the processes.

The answer to the second question is "yes"! If T_n is replaced by

$$\tilde{T}_n = \int_{-\infty}^{\infty} |\sqrt{n}(\mathbb{F}_n(x) - F_0(x)|d\mathbb{F}_n(x)$$

then, using a constructed version of $\mathbb{U}_n$ and $\mathbb{U}$ for which $\|\mathbb{U}_n - \mathbb{U}\|_\infty \rightarrow_p 0$ holds, then we have

$$\tilde{T}_n =_d \int_{-\infty}^{\infty} |\mathbb{U}_n(F_0(x))| d\mathbb{G}_n(F_0(t)) = \int_0^1 |\mathbb{U}_n(t)| d\mathbb{G}_n(t)$$

and we can write

$$\begin{aligned} |\tilde{T}_n - T| &\leq |\int_0^1 |\mathbb{U}_n(t)| d\mathbb{G}_n(t) - \int_0^1 |\mathbb{U}(t)| dt| \\ &\leq |\int_0^1 |\mathbb{U}_n(t)| d\mathbb{G}_n(t) - \int_0^1 |\mathbb{U}(t)| d\mathbb{G}_n(t)| \\ &\qquad + |\int_0^1 |\mathbb{U}(t)| d\mathbb{G}_n(t) - \int_0^1 |\mathbb{U}(t)| dt| \\ &\leq \|\mathbb{U}_n - \mathbb{U}\|_\infty + |\int_0^1 |\mathbb{U}(t)| d(\mathbb{G}_n(t) - t)| \\ &\rightarrow_p 0 + 0 \end{aligned}$$

where the second term converges to zero a.s. since $\mathbb{U}(t, \omega)$ is a bounded continuous function for almost every ω (for all ω if we construct $\mathbb{U}$ "correctly"), and by the Helly-Bray theorem since $\mathbb{G}_n \rightarrow I$ with probability 1.

5. (42 points). Suppose that B is standard Brownian motion on $[0, \infty)$.
A. Since $f(x) = x^2$ is convex and B_t is a martingale, B_t^2 is a submartingale. What do we subtract from B_t^2 to get a martingale? Justify your answer.
B. The Ito calculus says that for twice differentiable functions f from R to R we have

$$f(B_t) - f(0) = \int_0^t f'(B_s) dB_s + \text{something}. \tag{3}$$

What is "something" in (3)? Hint: Note that "something" should reduce to your answer in part A when $f(x) = x^2$.
C. Apply (3) to the function $f(x) = \sinh(x)$ (recalling that $\sinh(x) \equiv (e^x - e^{-x})/2$).

D. Using the result from C, write down a martingale related to $\sinh(B_t)$.

Solution: A. Subtracting t from B_t^2 yields a martingale: by straightforward calculation

$$\begin{aligned} E(B_t^2 - t|\mathcal{A}_s) &= E\{(B_t - B_s + B_s)^2 - [(t-s)+s]|\mathcal{A}_s\} \\ &= E\{(B_t - B_s)^2 + 2(B_t - B_s)B_s + B_s^2 - [(t-s)+s]|\mathcal{A}_s\} \\ &= E\{(B_t - B_s)^2\} - (t-s) \\ &\qquad + 2B_s E\{(B_t - B_s)|\mathcal{A}_s\} + B_s^2 - s \\ &\qquad \text{since } B_t - B_s \text{ is independent of } \mathcal{A}_s \\ &= 0 + 0 + B_s^2 - s \qquad a.s. \end{aligned}$$

Thus $B_t^2 - t$ is a martingale.

B. The Ito calculus says that for twice differentiable functions f from R to R we have

$$f(B_t) - f(0) = \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds\,. \tag{4}$$

Thus "something" in (3) is just

$$\frac{1}{2}\int_0^t f''(B_s)ds\,.$$

Note that for $f(x) = x^2$ we have $f'(x) = 2x$, $f''(x) = 2$, and the "something" term becomes

$$\frac{1}{2}\int_0^t 2ds = t\,.$$

Thus the Ito calculus yields

$$B_t^2 = 2\int_0^t B_s dB_s + t$$

or equivalently

$$B_t^2 - t = 2\int_0^t B_s dB_s$$

is a martingale.

C. For the function $f(x) = \sinh(x)$, $f'(x) = \cosh(x)$, $f''(x) = \sinh(x)$, and since $\sinh(0) = 0$ the Ito formula yields

$$\sinh(B_t) = \int_0^t \cosh(B_s) dB_s + \frac{1}{2} \int_0^t \sinh(B_s) ds\,.$$

D. It follows from the calculation of C that

$$\sinh(B_t) - \frac{1}{2} \int_0^t \sinh(B_s) ds = \int_0^t \cosh(B_s) dB_s$$

is a martingale.

6. (42 points). Prove *one* of the following two inequalities:

$$P(|X| \geq 1/\epsilon) \leq \frac{7}{\epsilon} \int_0^\epsilon (1 - \text{Real}\phi(t)) dt\,;$$

$$P(|X| \geq 1/\epsilon) \leq \frac{1}{2\epsilon} \int_{[|t| \leq 2\epsilon]} |1 - \phi(t)| dt\,. \tag{5}$$

Hints: In proving the first inequality, you may use the fact that $\inf_{|y| \geq 1}(1 - \sin(y)/y) = (1 - \sin(1)) = .1585\ldots \geq 1/7$. The second inequality was proved in the solution to problem set 3, problem 2.

Solution: The first inequality is proved in Section 13.3, page 293, PfS. The second inequality is proved as follows: first note that for $T \in (0, \infty)$ we have, by Fubini's theorem,

$$\begin{aligned}
\frac{1}{2T} \int_{-T}^{T} \phi(t) dt &= \frac{1}{2T} \int_{-T}^{T} E(\cos(tX) + i \sin(tX))\, dt \\
&= \frac{1}{2T} E\left\{ \int_{-T}^{T} (\cos(tX) + i \sin(tX))\, dt \right\} \\
&= E\left(\frac{\sin(TX)}{TX} \right)\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
|\frac{1}{2T}\int_{-T}^{T}\phi(t)dt| &\le E|\frac{\sin(TX)}{TX}| \\
&\le E|\frac{\sin(TX)}{TX}|1_{[|X|\ge\epsilon]} + E|\frac{\sin(TX)}{TX}|1_{[|X|<\epsilon]} \\
&\le \frac{1}{T\epsilon}P(|X|\ge\epsilon) + 1 - P(|X|\ge\epsilon)
\end{aligned}$$

since $|\sin(y)|\le 1$ and $|\sin(y)/y|\le 1$. Choosing $T=2/\epsilon$ yields

$$|\frac{\epsilon}{4}\int_{-2/\epsilon}^{2/\epsilon}\phi(t)dt| \le 1 - \frac{1}{2}P(|X|\ge\epsilon)$$

or, equivalently,

$$\begin{aligned}
P(|X|\ge\epsilon) &\le 2 - |\frac{\epsilon}{2}\int_{-2/\epsilon}^{2/\epsilon}\phi(t)dt| \\
&= \frac{\epsilon}{2}\int_{|t|\le 2/\epsilon}dt - |\frac{\epsilon}{2}\int_{-2/\epsilon}^{2/\epsilon}\phi(t)dt| \\
&\le \frac{\epsilon}{2}\int_{-2/\epsilon}^{2/\epsilon}|1-\phi(t)|dt\,;
\end{aligned}$$

i.e. (5) holds.

7. (42 points). Suppose that you are given the law of the iterated logarithm for Brownian motion $\mathbb{S}$ at ∞:

$$\limsup_{t\to\infty}\frac{\mathbb{S}(t)}{\sqrt{2t\log\log t}} = 1 \qquad a.s. \tag{6}$$

A. Prove the *time reversal* property of Brownian motion: if $\mathbb{S}$ is standard Brownian motion, then the process $\tilde{\mathbb{S}}(t)\equiv t\mathbb{S}(1/t)$ is also standard Brownian motion.

B. Use A together with (6) to prove the LIL for Brownian motion at 0:

$$\limsup_{t\to 0}\frac{\mathbb{S}(t)}{\sqrt{2t\log\log(1/t)}} = 1 \qquad a.s. \tag{7}$$

Solution: A. $\tilde{\mathbb{S}}(t) \equiv t\mathbb{S}(1/t)$ is clearly a mean zero Gaussian process because $\mathbb{S}$ is. Furthermore for $0 \le s, t < \infty$ we have

$$\begin{aligned} E(\tilde{\mathbb{S}}(s)\tilde{\mathbb{S}}(t)) &= E(st\mathbb{S}(1/s)\mathbb{S}(1/t)) \\ &= st\,\{(1/s) \wedge (1/t)\} \\ &= s \wedge t\,. \end{aligned}$$

Thus $\tilde{\mathbb{S}}$ is standard Brownian motion.

B. For $1/s = t$ we have

$$\begin{aligned} \frac{\mathbb{S}(t)}{\sqrt{2t\log\log t}} &= \frac{\mathbb{S}(1/s)}{\sqrt{2(1/s)\log\log(1/s)}} \\ &= \frac{s\mathbb{S}(1/s)}{\sqrt{2s\log\log(1/s)}} \\ &= \frac{\tilde{\mathbb{S}}(s)}{\sqrt{2s\log\log(1/s)}}\,. \end{aligned}$$

where $\mathbb{S}(t) \equiv t\mathbb{S}(1/t)$. It follows that

$$\begin{aligned} 1 &= \limsup_{t\to\infty} \frac{\mathbb{S}(t)}{\sqrt{2t\log\log t}} \\ &= \limsup_{s\to 0} \frac{\tilde{\mathbb{S}}(s)}{\sqrt{2s\log\log(1/s)}} \\ &= \limsup_{s\to 0} \frac{\mathbb{S}(s)}{\sqrt{2s\log\log(1/s)}} \end{aligned}$$

where the last line follows from the result of A. Thus the LIL for Brownian motion at 0 follows from the LIL for Brownian motion at ∞ via time inversion.