Statistics 523, Problem Set 7
Wellner; 5/12/99

Reading: Shorack, PfS; Chapter 18, pages 395-448.
(continuous parameter parts!).

Due: Wednesday, May 19, 1999.

1. Use a reflection principle to show that for $0 \le y \le x$

$$P(\sup_{0\le s\le t} \mathbb{S}(s) \ge x,\ \mathbb{S}(t) \le y) = P(\mathbb{S}(t) \ge 2x - y)\,,$$

and use this to show that the joint density of $M^+ \equiv \sup_{0\le s\le t} \mathbb{S}(s), \mathbb{S}(t)$ is given by

$$f(x,y) = \sqrt{\frac{2}{\pi t^3}}(2x-y)\exp\left(-\frac{(2x-y)^2}{2t}\right) \qquad \text{for } \ 0 \le y \le x\,.$$

2. **Bonus Problem 1.** Suppose that $\mathbb{S}$ is standard Brownian motion on $(C[0,\infty), \mathcal{C}_{[0,\infty)})$, and let its distribution be denoted by $P = P_0$. Let $\mathbb{S}_\mu(t) \equiv \mathbb{S}(t) + \mu t$ be Brownian motion with drift μ, and let P_μ denote the distribution of $\mathbb{S}_\mu$ on $(C[0,\infty), \mathcal{C}_{[0,\infty)})$. Set $Y(t) \equiv \exp(\mu\mathbb{S}(t) - \mu^2 t/2)$. For $t > 0$ let $P_{0,t}$ and $P_{\mu,t}$ denote the distributions P_0 and P_μ restricted to $\mathcal{A}_t \equiv \{\mathbb{S}(s) : s \le t\}$. Show that the Radon - Nikodym derivative $dP_{\mu,t}/dP_{0,t} = Y(t)$.

3. **Bonus Problem 2.** Suppose that $\mathbb{S}_\mu$ is Brownian motion with drift $\mu > 0$ as in problem 2, and let $\tau \equiv \inf\{t > 0 : \mathbb{S}_\mu(t) = a\}$, $a > 0$. Use the result of bonus problem 1 together with results from problem set 6 to find the distribution of τ. You should find that

$$\begin{aligned} P_\mu(\tau > t) &= P_\mu(\mathbb{S}_\mu(s) < a,\ 0 \le s \le t) \\ &= \Phi\left(\frac{a-\mu t}{\sqrt{t}}\right) - e^{2\mu}\Phi\left(\frac{-a-\mu t}{\sqrt{t}}\right) \end{aligned}$$

and

$$f_\tau(t) = \frac{a}{\sqrt{2\pi t^3}}\exp\left(-\frac{(a-\mu t)^2}{2t}\right) \qquad \text{for } t \ge 0\,.$$

This is the *inverse Gaussian* density. [Note that this reduces to the density of τ_a from problem set 6 when $\mu = 0$.