Statistics 523, Problem Set 2

Wellner; 4/7/99

Reading: Shorack, PfS; Chapter 14, pages 307-334;
Chapter 15, pages 335 - 348.

Due: Wednesday, April 14, 1999.

1. PfS, Exercise 14.1.2, page 309.
2. PfS, Exercise 14.1.4, page 309.
3. PfS, Exercise 14.2.9, page 318.
4. Suppose that $X_{n1}, \ldots, X_{nn}$ are independent random variables with $X_{nk} \sim$ Bernoulli(p_{nk}), and let $Y_{nk} \sim$ Poisson(p_{nk}) for $k = 1, \ldots, n$. Let P_n be the distribution of $X_n \equiv \sum_{k=1}^n X_{nk}$ and let Q_n be the distribution of $Y_n \equiv \sum_{k=1}^n Y_{nk}$. Show that
$$d_{TV}(P_n, Q_n) \equiv \sup\{|P_n(A) - Q_n(A)| : \; A \in \mathcal{B}\} \le \sum_{k=1}^n p_{nk}^2 \, .$$
Note that if $p_{nk} = \lambda_k / n$ for $k = 1, \ldots, n$, then the bound becomes $\bar{\lambda}/n$.
[Hint: Construct X_n and Y_n on a common probability space as follows: Let $T_{nk} \sim$ Poisson(p_{nk}), $k = 1, \ldots, n$ and $Z_{nk} \sim$ Bernoulli$(1 - (1 - p_{nk})e^{p_{nk}})$, $k = 1, \ldots, n$ all be independent, and define
$$X_{nk} = 1_{[T_{nk} \ge 1]} + 1_{[T_{nk} = 0]} 1_{[Z_{nk} = 1]} \, .$$
Set $X_n \equiv \sum_{k=1}^n X_{nk}$ and $Y_n = \sum_{k=1}^n T_{nk}$. Check that $X_{nk} \sim$ Bernoulli(p_{nk}),
$$\begin{aligned} P(T_{nk} = 0, X_{nk} = 1) &= e^{-p_{nk}} - (1 - p_{nk}) \, , \\ P(T_{nk} \ge 1, X_{nk} = 0) &= 0 \, , \\ P(T_{nk} \ge 2) &= 1 - e^{-p_{nk}} - p_{nk} e^{-p_{nk}} \, . \end{aligned}$$
Show that
$$d_{TV}(P_n, Q_n) \le P(X_n \ne Y_n) \le \sum_{k=1}^n P(X_{nk} \ne T_{nk}) \le \sum_{k=1}^n p_{nk}^2 \, .]$$

5. **Bonus Problem:** PfS, Exercise 14.1.1, page 309.