Statistics 523, Problem Set 2 Solutions

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1. Exercise 12.9.3, PfS (2017) page 328: Prove theorem 9.1(b) when all $t_{n,k} = k/2^n$. [Hint: Use the Paley-Zygmund inequality.]

   Solution: First note that since the partitions are nested ($\mathcal{P}_n \subset \mathcal{P}_{n+1}$), $V_n(1)$ is non-decreasing in n with probability 1: i.e. $V_n(1) \le V_{n+1}(1)$ for all n with probability 1.

   By the Paley-Zygmund inequality, for any $0 < \lambda < 1$,

   $$P(V_n(1) > \lambda E V_n(1)) \ge (1-\lambda^2) E^2 V_n(1)/E(V_n^2(1)). \tag{1}$$

   So we need to compute $EV_n(1)$ and $E[V_n^2(1)]$. But $\mathbb{S}(k/2^n) - \mathbb{S}((k-1)/2^n) \stackrel{d}{=} 2^{-n/2} Z$ where $Z \sim N(0,1)$. Thus

   $$EV_n(1) = \sum_{k=1}^{2^n} E|\mathbb{S}(k/2^n) - \mathbb{S}((k-1)/2^n)| = 2^n \cdot 2^{-n/2} E|Z| = 2^{n/2} E|Z| \to \infty.$$

   On the other hand

   $$\begin{aligned} EV_n^2(1) &= E\left\{ \sum_{k=1}^{2^n} |\mathbb{S}(k/2^n) - \mathbb{S}((k-1)/2^n)| \cdot \sum_{j=1}^{2^n} |\mathbb{S}(j/2^n) - \mathbb{S}((j-1)/2^n)| \right\} \\ &= 2^n(2^n-1)2^{-n} E|Z_k| E|Z_j| 1\{k \ne j\} + 2^n \cdot 2^{-n} E|Z_k|^2 \\ &= (2^n - 1)\left(E|Z|\right)^2 + 1. \end{aligned}$$

   Thus the factor on the right side of (1) is

   $$\frac{[EV_n(1)]^2}{E(V_n^2(1))} = \frac{2^n (E|Z|)^2}{(2^n-1)(E|Z|)^2 + 1} \to 1 \tag{2}$$

   as $n \to \infty$. Thus the right side of (1) converges to $(1-\lambda)^2$. Now fix $M > 0$ large and $\lambda > 0$ small. Since $EV_n(1) \to \infty$, we can choose $n \ge N_{\lambda,M}$ so that $\lambda EV_n(1) \ge M$ and the left side of (2) is $> 1-\lambda$. Thus $P(V_n(1) > M) \ge (1-\lambda)^3$ for $n \ge N_{\lambda,M}$. Since $V_n(1) \nearrow$ a.s., this implies that $P(\lim_n V_n(1) > M) \ge (1-\lambda)^3$. But since M is arbitrarily large and λ is arbitrarily small this yields $V_n(1) \to_{a.s.} \infty$.

2. Suppose that $\xi_1, \xi_2, \ldots, \xi_n$ are i.i.d. Uniform$(0,1)$ rv's, let $\xi_{n:0} \equiv 0 \le \xi_{n:1} \le \xi_{n:2} \le \cdots \le \xi_{n:n} \le 1 \equiv \xi_{n:n+1}$ be the order statistics, and let $\delta_{n:i} \equiv (\xi_{n:i} - \xi_{n:i-1})$, $i \in \{1, \ldots, n+1\}$, be the *spacings*. Give a direct proof of the fact that

$\sqrt{n}\max_{1\le i\le n+1}\delta_{n:i}\to_p 0$. For what sequences $c_n \nearrow \infty$ do we have $c_n\max_{1\le i\le n+1}\delta_{n:i} = O_p(1)$?

Solution: (a) Note that for $t>0$

$$\begin{aligned} P(\sqrt{n}\max_{1\le i\le n+1}\delta_{n:i}>t) &= P(\max_{1\le i\le n+1}\delta_{n:i}>t/\sqrt{n}) \\ &\le \sum_{i=1}^{n+1}P(\delta_{n:i}>t/\sqrt{n}) = (n+1)P(\delta_{n:1}>t/\sqrt{n}) \\ &= (n+1)P(\xi_{n:1}>t/\sqrt{n}) = (n+1)\left(1-\frac{t}{\sqrt{n}}\right)^n \\ &\le (n+1)\{\exp\left(-t/\sqrt{n}\right)\}^n \\ &= (n+1)\exp(-\sqrt{n}t) \end{aligned}$$

where we used $1-x\le e^{-x}$ for $x\ge 0$ to get the second inequality. This bound converges to 0 for any small $t>0$, and hence $\sqrt{n}\max_{1\le i\le n+1}\delta_{n:i}\to_p 0$.
(b) If we replace $\sqrt{n}$ by $b_n\to\infty$, then the same argument yields

$$P(b_n\max_{1\le i\le n+1}\delta_{n:i}>t)\le (n+1)\exp\left(-(n/b_n)t\right).$$

Then with $b_n = c^{-1}n/\log n$ the bound becomes

$$(n+1)\exp\left(-ct\log n\right) = (n+1)n^{-ct}\to 0$$

if $ct>1$. Thus with $c=1$, $\max_{1\le i\le n+1}\delta_{n:i} = O_p(n^{-1}\log n)$. In fact, it is known (Darling, Lévy; see Shorack and W (1986), page 726) that

$$(n+1)\max_{1\le i\le n+1}\delta_{n:i}-\log(n+1)\to_d Y$$

where $P(Y\le t)=\exp(-e^{-t})$.

3. Suppose that $Y_1, Y_2, \ldots, Y_{n+1}$ are i.i.d. Exponential (1) random variables, and let $S_k \equiv \sum_{j=1}^k Y_j$ for $1\le k\le n+1$ In the Skorokhod embedding theorem proof it was claimed that

$$\left(\frac{S_1}{S_{n+1}}, \frac{S_2}{S_{n+1}}, \ldots, \frac{S_n}{S_{n+1}}\right) \stackrel{d}{=} (\xi_{n:1},\ldots,\xi_{n:n}).$$

Prove this.

Proof: By independence, the joint density of the Y_j's is simply the product of the marginal densities:

$$f_{\underline{Y}}(\underline{y}) = \prod_{j=1}^{n+1} e^{-y_j} = e^{-\sum_{j=1}^{n+1}y_j}1_{[y_1>0,\ldots,y_{n+1}>0]}.$$

Let $(S_1, S_2, \ldots, S_{n+1})$ with $S_k \equiv \sum_{j=1}^{k}$ for $1 \le k \le n+1$. Then $Y_k = S_k - S_{k-1}$ for $1 \le k \le n+1$ and hence the Jacobian of the transformation is 1. Thus the joint density of $(S_1, S_2, \ldots, S_{n+1})$ is given by

$$f_{\underline{S}}(s_1, \ldots, s_{n+1}) = e^{-s_{n+1}} 1_{[0<s_1<\cdots<s_{n+1}]}.$$

Now set $Z_k \equiv S_k / S_{n+1}$ for $1 \le k \le n$ and $W \equiv S_{n+1}$. The inverse transformation is given by $S_k = Z_k W$ for $1 \le k \le n+1$ ($Z_{n+1} \equiv 1$) and has Jacobian w^n, so the joint density of $(S_1, \ldots, S_n, W)$ is given by

$$\begin{aligned} g_{(\underline{Z}, W)}(z_1, \ldots, z_n, w) &= e^{-w} w^n 1_{[0<z_1<\cdots<z_n<1,\ w>0]} \\ &= \frac{w^n}{n!} e^{-w} 1_{[0,\infty)}(w) \cdot n! 1_{[0<z_1<\cdots<z_n<1]}. \end{aligned}$$

Thus $W \sim \text{Gamma}(n+1, 1)$ and $(Z_1, \ldots, Z_n) \stackrel{d}{=} (\xi_{n:1}, \ldots, \xi_{n:n})$ are independent where $(\xi_{n:1}, \ldots, \xi_{n:n})$ are the order statistics of n independent $\text{Uniform}(0,1)$ random variables $\xi_1, \ldots, \xi_n$.

4. Suppose that $g(x) = \int_0^1 f_0(t) x(t) dt$ for some fixed absolutely continuous function f_0 on $[0,1]$ with $\int_0^1 [f_0'(t)]^2 dt < \infty$. What does Strassen's theorem say about $\limsup_{n\to\infty} g(\mathbb{Z}_n)$ where $\mathbb{Z}_n \equiv \mathbb{S}(n\cdot)/\sqrt{2n \log\log n}$ and $\mathbb{S}$ is a fixed Brownian motion process on $[0,\infty)$.
Solution: Strassen's theorem implies that

$$\limsup_{n\to\infty} g(\mathbb{Z}_n) = \sup_{f \in \mathcal{K}} g(f) = \sup_{f \in \mathcal{K}} \int_0^1 f_0(t) f(t) dt.$$

To compute the supremum on the right side in the last display, note that

$$\begin{aligned} |g(f)| &= \Big| \int_0^1 f_0(t) f(t) dt \Big| \\ &= \Big| \int_0^1 f_0(t) \int_0^t f'(s) ds dt \Big| = \Big| \int_0^1 \int_0^1 1_{[0,t]}(s) f_0(t) f'(s) ds dt \Big| \\ &= \Big| \int_0^1 f'(s) \left(\int_0^1 1_{[s,1]}(t) f_0(t) dt \right) ds \Big| = \Big| \int_0^1 f'(s) h_0(s) ds \Big| \end{aligned}$$

where $h_0(s) \equiv \int_s^1 f_0(t) dt$. But then it follows by the Cauchy-Schwarz inequality that

$$\begin{aligned} |g(f)| &= \Big| \int_0^1 f'(s) h_0(s) ds \Big| \le \left(\int_0^1 [f'(s)]^2 ds \right)^{1/2} \cdot \left(\int_0^1 [h_0(s)]^2 ds \right)^{1/2} \\ &\le 1 \cdot \left(\int_0^1 [h_0(s)]^2 ds \right)^{1/2} \equiv \|h_0\|_2 \end{aligned}$$

with equality in the inequality if $f'(s) = ch_0(s)$ for some constant c, and if c is chosen so that $1 = \int_0^1 [f'(s)]^2 ds = c^2 \int_0^1 [h_0(s)]^2 ds$, which implies that $c = 1/\|h_0\|_2$. The upshot is that

$$\sup_{f \in \mathcal{K}} g(f) = \|h_0\|_2$$

with equality if $f'(s) = h_0(s)/\|h_0\|_2$, and hence $f(t) = \int_0^t h_0(s) ds / \|h_0\|_2$.

Here is an interesting special case: if $f_0(t) \equiv 1$ for $0 \le t \le 1$, then $h_0(t) = 1 - t$, and $\|h_0\|_2^2 = \int_0^1 (1-t)^2 dt = 1/3$. This is connected to our result that $\int_0^1 \mathbb{S}(t) dt \sim N(0, 1/3)$.

5. PfS (2017), Exercise 9.3.6; page 205.
Let ϕ be a chf. Show that $c^{-1} \int_0^c \phi(tu) du$ is a chf.

Solution: Let U be a Uniform$(0, c)$ random variable independent of X. Then let $Y \equiv UX$. The characteristic function of Y is

$$\begin{aligned} \phi_Y(t) &= Ee^{itY} = Ee^{itUX} = E\{E\{e^{itUX}|U\}\} \\ &= E\{\phi_X(tU)\} = \frac{1}{c}\int_0^c \phi_X(tu) du. \end{aligned}$$

Thus if ϕ is the characteristic function of X, then the given expression is the characteristic function of UX where $U \sim$ Uniform$(0, c)$ is independent of X.

In fact a random variable Y has a unimodal distribution if and only if it has a characteristic function of the form given in the display with $c = 1$ (and hence also if and only if $Y = UX$ for $U \sim$ Uniform$(0, 1)$ and $X \sim F$); see e.g. Dharmadhikari,and Joag-Dev (1988), *Unimodality, Convexity, and Applications*, page 7.

6. Bonus question 1: PfS (2017), Exercise 9.3.4, page 205).
Derive the Logistic$(0, 1)$ characteristic function. Hint: use lemma 9.3.2.

Solution: The logistic density is given by $f(x) = e^{-x}/(1 + e^{-x})^2$, so we want to calculate

$$\phi(t) = Ee^{itX} = \int_{-\infty}^{\infty} \frac{e^{itx}e^{-x}}{(1 + e^{-x})^2} dx.$$

We want to show that this equals $\pi t / \sinh(\pi t)$ for $t \in \mathbb{R}$. Consider the function ϕ as a function of a complex variable z: thus

$$\phi(z) = \int_{-\infty}^{\infty} \frac{e^{izx}e^{-x}}{(1 + e^{-x})^2} dx.$$

Also consider the function $\psi(z) = \pi z/\sinh(\pi z)$. Now both ϕ and ψ are analytic functions of $z = x + iy$ for $|Im(z)| < 1$. Note that

$$\psi(iy) = \frac{\pi(iy)}{\sinh(\pi(iy))} = \frac{\pi iy}{i\sin(\pi y)} = \frac{\pi y}{\sin(\pi y)}$$

so that when $y = \pm 1$,

$$\psi(\pm i) = \frac{\pm\pi}{\sin(\pm\pi)} = \pm\infty.$$

But they are both analytic on the strip $D = \{z : \; |Im(z)| < 1\}$. For ϕ on the imaginary axis we use the change of variables $(1 + e^{-x}) = v$, so that $e^{-x} = (1-v)/v$, to find that

$$\begin{aligned}
\phi(iy) &= \int_{-\infty}^{\infty} \frac{e^{-yx}e^{-x}}{(1+e^{-x})^2} dx = \int_{-\infty}^{\infty} e^{-yx} d\{(1+e^{-x})^{-1}\} \\
&= \int_0^1 \left(\frac{1-v}{v}\right)^y dv = \int_0^1 v^{-y}(1-v)^y dv = \frac{\Gamma(1-y)\Gamma(y+1)}{\Gamma(2)} \\
&= \Gamma(1-y)\Gamma(y)y = \frac{\pi y}{\sin(\pi y)} = \psi(iy)
\end{aligned}$$

for $y \in (-1, 1)$. Here we used the "duplication formula for the Gamma function",

$$\Gamma(1-y)\Gamma(y) = \frac{\pi}{\sin(\pi y)},$$

in the last step. Since $S = \{z : \; z = iy, \; |y| < 1\}$ has an accumulation point in D, it follows from Lemma 9.3.2 that $\phi(z) = \psi(z)$ for $z \in D$. But this implies that $\psi(t) = \phi(t)$ for $t \in \mathbb{R}$; i.e. the chf ϕ of the logistic distribution is $\pi t/\sinh(\pi t)$ for $t \in \mathbb{R}$.