Statistics 523, Midterm Exam Solutions

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1. (30 points) Suppose that ϕ is the characteristic function (of some random variable X). Show that the real part of ϕ (or $Re\phi$) is also a characteristic function.

   Solution: Note that since $\overline{\phi}_X = \overline{\phi}_{-X}$ we have

$$
\begin{aligned}
\mathrm{Re}\phi_X(t) &= \frac{1}{2}(\phi_X(t) + \overline{\phi}_X(t)) = \frac{1}{2}(\phi_X(t) + \phi_{-X}(t)) \\
&= \frac{1}{2}\left(Ee^{itX} + Ee^{-itX}\right) = \frac{1}{2}\left(\int e^{itx} dF_X(x) + \int e^{itx} dF_{-X}(x)\right) \\
&= \int e^{itx} d(F_X(x) + F_{-X}(x))/2 \equiv \int e^{itx} dG(x)
\end{aligned}
$$

   where $G(x) \equiv (1/2)(F_X(x) + F_{-X}(x))$ is the distribution function of ϵX where ϵ is a Rademacher random variable independent of X.

2. (48 points)
   A. Suppose that X has $E(X) = 0$ and $Var(X) = \sigma^2 < \infty$, and write X^* for a random variable having the $X-$zero bias distribution satisfying $\sigma^2 Ef'(X^*) \ = \ E[Xf(X)]$.
   (i) Show that for any real number $a \neq 0$ we have $(aX)^* \stackrel{d}{=} aX^*$.
   (ii) Show that if $|X| \le C$ for some constant C then $|X^*| \le C$.
   B. Now suppose that X is a non-negative random variable with mean $\mu = E(X)$, and write X^s for a random variable having the X-size bias distribution (function) F^s satisfying $E[Xf(X)] = \mu Ef(X^s)$ for all functions f for which $E[Xf(X)]$ exists.
   (i) Show that for any real number $a > 0$ we have $(aX)^s \stackrel{d}{=} aX^s$.
   (ii) Show that if $0 \le X \le C$ for some constant C then $0 \le X^s \le C$.

   Solution: In connection with part A of this problem, it should be noted that (2.53) in C-G-S should be replaced by $\sigma^2 Ef'(X^*) = E[Xf(X)]$.
   A(i) Let $\sigma^2 = Var(X)$ and define $g(x) \equiv f(ax)$ so that $g'(x) = af'(ax)$,

we have

$$\begin{aligned}(a\sigma)^2 Ef'(aX^*) &= a\sigma^2 Eg'(X^*) = aE(Xg(X)) \\ &= E[(aX)f(aX)] = (a\sigma)^2 Ef'\left((aX)^*\right)\end{aligned}$$

so we conclude that $(aX)^* \stackrel{d}{=} aX^*$.
B(i) Let $\mu = E(X)$ and define $g(x) \equiv f(aX)$. Then

$$\begin{aligned}a\mu Ef(aX^s) &= a\mu Eg(X^s) = aE(Xg(X)) \\ &= E[(aX)f(aX)] = a\mu Ef((aX)^s))\end{aligned}$$

so we conclude that $(aX)^s \stackrel{d}{=} aX^s$.
A(ii) Suppose that $|X| \le C$ almost surely. Then let $f(x) = (x - C)1_{[X>C]} + (x+C)1_{[X<-C]}$. Thus f is absolutely continuous with $f'(x) = 1_{[|x|>C]}$. Thus from the definition of the X^*-zero bias transformation

$$\sigma^2 P(|X|^* > c) = \sigma^2 Ef'(X^*) = EXf(x) = 0$$

since $f(X) = 0$ for $X \in (-C, C)$.
B(ii) Suppose that $X \in [0, C]$ almost surely. Then consider $f(x) \equiv 1_{[0,C]^c}(x)$. But then by the fundamental equation for size-biased sampling

$$\mu P(X^s \in [0, C]^c) = \mu Ef(X^s) = E[Xf(X)] = 0$$

since $f(X) = 0$ for $X \in [0, C]^c$. Thus $X^* \in [0, C]$ almost surely.

3. (36 points) If X is a random variable with $E(X) = 0$ and $Var(X) = \sigma^2 < \infty$, then the density f^* of the $X-$zero biased distribution F^* exists and is given by (2.54), page 27, of C-G-S (2011). (They call it p^*.)
(i) Show that if the distribution function F has density f, then $f^* \equiv p^*$ is unimodal, with mode at 0.
(ii) What can you say if F is arbitrary (with mean 0 and finite variance)?

Solution: (i) First suppose that the distribution function F of X is absolutely continuous with density f. Then

$$f^*(x) = E[X1_{(x,\infty)}(X)]/\sigma^2 = \int_{(x,\infty)} xf(x)dx/\sigma^2,$$

and hence

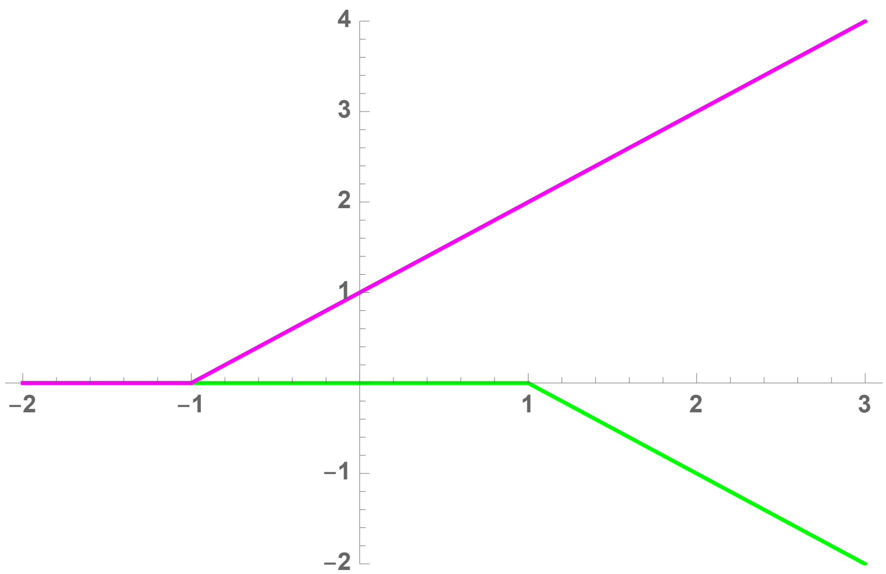
$$\frac{d}{dx} f^*(x) = -x f(x)/\sigma^2 \begin{cases} < 0, & x > 0 \\ \geq 0, & x \leq 0 \end{cases}$$

with equality at $x = 0$ if f is continuous at 0. This implies that f^* is unimodal with mode at 0.

(ii) If F is not absolutely continuous, consider the distribution function G^* of X^* given by (2.55), C-G-S page 27:

$$G^*(x) = E[X(X-x)1_{[X \leq x]}]/\sigma^2 = \int_{(-\infty, x]} y(y-x)dF(y)/\sigma^2.$$

We may assume without loss of generality that $\sigma^2 = 1$; if not replace X by X/σ with d.f. $\tilde{F}(x) = F(\sigma x)$. Now let $H(x, y) \equiv y(y-x)1_{[x-y \geq 0]}$. Note that H is a concave function of x for each fixed $y \geq 0$, and it is a convex function of x for each fixed $y \leq 0$; see the figure below.



$H-$function plots: y=-1 (magenta); y=+1 (green)

Thus if $\lambda \in [0, 1]$, $x_1, x_2 \in \mathbb{R}$, $y \geq 0$, and $\overline{\lambda} \equiv 1 - \lambda$,

$$H(\lambda x_1 + \overline{\lambda} x_2, y) \begin{cases} \geq \lambda H(x_1, y) + \overline{\lambda} H(x_2, y), & y \geq 0, \\ \leq \lambda H(x_1, y) + \overline{\lambda} H(x_2, y), & y \leq 0. \end{cases}$$

Now fix $x_1, x_2 \leq 0$ and replace y by $X \sim F$. Then since $\lambda x_1 + \overline{\lambda} x_2 \leq 0$, taking the expectation across the resulting inequality yields

$$\begin{aligned} G^*(\lambda x_1 + \overline{\lambda} x_2) &= E\{H(\lambda x_1 + \overline{\lambda} x_2, X)\} \\ &\leq \lambda EH(x_1, X) + \overline{\lambda} EH(x_2, X) \\ &= \lambda G^*(x_1) + \overline{\lambda} G^*(x_2); \end{aligned}$$

i.e. G^* is convex on $(-\infty, 0]$. Similarly, if we fix $x_1, x_2 \geq 0$ and replace Y by $X \sim F$, then since $\lambda x_1 + \overline{\lambda} x_2 \geq 0$ for $\lambda \in [0, 1]$, taking the expectation across the resulting inequality yields

$$\begin{aligned} G^*(\lambda x_1 + \overline{\lambda} x_2) &= E\{H(\lambda x_1 + \overline{\lambda} x_2, X) \\ &\geq \lambda EH(x_1, X) + \overline{\lambda} EH(x_2, X) \\ &= \lambda G^*(x_1) + \overline{\lambda} G^*(x_2); \end{aligned}$$

i.e. G^* is concave on $[0, \infty)$. But since G^* is convex on $(-\infty, 0]$ and concave on $[0, \infty)$, G^* is unimodal about 0 (according to the definition of Khintchine (1938); see e.g. Dharmadhikari and Joag-dev (1988), page 2).

4. (36 points) Suppose that $K(t) \equiv E\{X(1_{[0 \leq t \leq X]} - 1_{[X \leq t < 0]})\}$ where X is a random variable with $E(X) = 0$. Show that $\int_{\mathbb{R}} K(t)dt = E(X^2)$ and $\int_{\mathbb{R}} |t|K(t)dt = 2^{-1}E|X|^3$.

Solution: First,

$$\begin{aligned} \int_{-\infty}^{\infty} K(t)dt &= \int_{-\infty}^{0} K(t)dt + \int_{0}^{\infty} K(t)dt \\ &= \int_{-\infty}^{0} -E(X1_{[X \leq t < 0]})dt + \int_{0}^{\infty} E(X1_{[0 \leq t \leq X]})dt \\ &= -E[X \int_{-\infty}^{0} 1_{[X \leq t < 0]}dt] + E[X \int_{0}^{\infty} 1_{[0 \leq t \leq X]}dt] \\ &= E[X^2 1_{[X < 0]}] + E[X^2 1_{[X \geq 0]}] = E(X^2). \end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{-\infty}^{\infty} |t|K(t)dt &= \int_{-\infty}^{0} (-t)K(t)dt + \int_{0}^{\infty} tK(t)dt \\
&= \int_{-\infty}^{0} -E(X(-t)1_{[X\le t<0]}dt + \int_{0}^{\infty} E(Xt1_{[0\le t\le X]})dt \\
&= -E[X\int_{-\infty}^{0} t1_{[X\le t<0]}dt] + E[X\int_{0}^{\infty} t1_{[0\le t\le X]}dt] \\
&= E[\frac{1}{2}(-X^3)1_{[X<0]}] + E[\frac{X^3}{2}t\cdot 1_{[X\ge 0]}] = \frac{1}{2}E|X|^3.
\end{aligned}$$

5. (30 points) Suppose that g and h are non-decreasing functions from $\mathbb{R}$ to $\mathbb{R}$, and let X be a random variable satisfying $Eg^2(X) < \infty$ and $Eh^2(X) < \infty$. Show that $Cov[g(X), h(X)] \ge 0$. **Hint:** Let Y be an independent copy of X and consider

$$E(g(Y) - g(X))(h(Y) - h(X)).$$

Solution: Since g and h are non-decreasing, $(g(Y) - g(X))(h(Y) - h(X)) \ge 0$ almost surely. To see this, suppose that $X(\omega) \le Y(\omega)$. Then $g(Y(\omega)) \ge g(X(\omega))$ so $g(Y(\omega)) - g(X(\omega)) \ge 0$. On the other hand, if $X(\omega) \ge Y(\omega)$, then $g(Y(\omega)) - g(X(\omega)) \le 0$ and also $h(Y(\omega)) - h(X(\omega)) \le 0$, so $g(Y(\omega) - g(X(\omega))h(Y(\omega) - h(X(\omega)) \ge 0$. Thus we have, since $Y \overset{d}{=} X$ and Y and X are independent,

$$\begin{aligned}
0 &\le E(g(Y) - g(X))(h(Y) - h(X)) \\
&= Eg(Y)h(Y) + Eg(X)h(X) - Eg(X)h(Y) - Eg(Y)h(X) \\
&= 2\{Eg(Y)h(Y) - Eg(X)Eh(X)\} \\
&= 2Cov[g(X)h(X)].
\end{aligned}$$

Do either problem 6 or problem 7

6. (36 points). Suppose that you are given the law of the iterated logarithm for Brownian motion $\mathbb{S}$ at ∞:

$$\limsup_{t\to\infty} \frac{\mathbb{S}(t)}{\sqrt{2t\log\log t}} = 1 \qquad a.s. \tag{1}$$

(a) Prove the *time reversal* property of Brownian motion: if $\mathbb{S}$ is standard Brownian motion, then the process $\tilde{\mathbb{S}}(t) \equiv t\mathbb{S}(1/t)$ is also standard Brownian motion.
(b) Use (a) together with (1) to prove the LIL for Brownian motion at 0:

$$\limsup_{t\to 0} \frac{\mathbb{S}(t)}{\sqrt{2t\log\log(1/t)}} = 1 \qquad a.s. \tag{2}$$

Solution: (a) Since $\tilde{\mathbb{S}}$ is clearly a Gaussian process, it suffices to show that $E\tilde{\mathbb{S}}(t) = 0$, $Cov(\tilde{\mathbb{S}}(s), \tilde{\mathbb{S}}(t)) = s \wedge t$, and $\tilde{\mathbb{S}}(0) = 0$. But that $E\tilde{\mathbb{S}}(t) = tE\mathbb{S}(1/t) = 0$ for $t > 0$. Furthermore

$$\begin{aligned} E\tilde{\mathbb{S}}(s)\tilde{\mathbb{S}}(t) &= stE\{\mathbb{S}(1/s)\mathbb{S}(1/t)\} = st\left(\frac{1}{s} \wedge \frac{1}{t}\right) \\ &= st\left\{ \begin{array}{ll} (1/s) & s \ge t, \\ (1/t) & s \le t \end{array} \right\} = s \wedge t. \end{aligned}$$

Moreover $\tilde{\mathbb{S}}(0) = \lim_{t\searrow 0} t\mathbb{S}(1/t) = \lim_{v\nearrow\infty} v^{-1}\mathbb{S}(v) = 0$ a.s. by the strong law of large numbers.

(b) Since $t\mathbb{S}(1/t)$ is Brownian motion on $[0, \infty)$, (1) yields

$$\begin{aligned} 1 &=_{a.s.} \limsup_{t\to\infty} \frac{t\mathbb{S}(1/t)}{\sqrt{2t\log\log t}} \\ &= \limsup_{t\to\infty} \frac{\mathbb{S}(1/t)}{\sqrt{2(1/t)\log\log t}} \\ &= \limsup_{r\to 0} \frac{\mathbb{S}(r)}{\sqrt{2r\log\log(1/r)}}. \end{aligned}$$

7. (36 points). Suppose that $\mathbb{S}$ is standard Brownian motion on $[0, \infty)$, and define $\mathbb{Y}(t) \equiv e^{-t}\mathbb{S}(e^{2t})$ for $t \in \mathbb{R}$.
(a) Compute $E(\mathbb{Y}(t))$ and $Var(\mathbb{Y}(t))$ for $t \in \mathbb{R}$.
(b) Compute $Cov(\mathbb{Y}(s), \mathbb{Y}(t))$ for $s, t \in \mathbb{R}$.
(c) What is the joint distribution of $(\mathbb{Y}(s), \mathbb{Y}(t))$?
(d) Show that $\mathbb{Y}$ is a stationary process.
(e) Is there a connection between $\mathbb{Y}$ and a Brownian bridge process $\mathbb{U}$ (perhaps divided by $\sqrt{t(1-t)}$)?

8. **Bonus Problem:** (40 points)
Find necessary and sufficient conditions for the CLT in sampling without replacement. One way of proceeding might be to specialize the hypotheses of the theorem of Hájek (1961) to the special case of sampling without replacement in which $a_{i,j} = b_i c_j$ for arbitrary (distinct) numbers $\{c_1, \ldots, c_N\}$ and $b_1 = b_2 = \ldots = b_n = 1$, $b_{n+1} = \cdots = b_N = 0$ where $1 \le n \le N$. Then with $\pi = (\pi_1, \ldots, \pi_N)$ a random permutation of $(1, \ldots, N)$, the sum

$$Y = \sum_{i=1}^{N} b_i c_{\pi(i)}$$

$\overset{d}{=}$ the sum of the numbers drawn in sampling n balls without replacement from an urn containing N balls numbered by the c_j's .