

### Statistics 523, Problem Set 3

Wellner; 4/15/2020

**Reading:** Shorack, PfS (2017), Chapter 10, pages 225-247  
Goldstein, *Amer. Math. Monthly* **116** (2009), 45 - 60.

**Due:** Wednesday, April 22, 2020.

- (a) Give an example of a random variable  $Y$  with distribution function  $F$  on  $\mathbb{R}^+ = [0, \infty)$  for which  $EY^r = \infty$  for all  $r > 0$ .  
(b) Does your example in (a) have  $Eg(Y) < \infty$  for some measurable function  $g$  with  $g(y) \rightarrow \infty$  as  $y \rightarrow \infty$ ?

- Stein's method for convergence in distribution to the Poisson distribution depends on the following characterization:  $X \sim \text{Poisson}(\lambda)$  if and only if

$$E[Xf(X)] = \lambda E[f(X+1)]$$

for all functions  $f$  for which the expectations exist. Show that if  $X \sim \text{Poisson}(\lambda)$  then the identity in the display holds for any bounded function  $f$ .

- PfS, Exercise 10.1.5, page 228. Use the multivariate CLT to show that  $\mathbb{U}_n \rightarrow_{f.d.} \mathbb{U}$  where  $\mathbb{U}$  is a standard Brownian bridge process on  $[0, 1]$ .
- PfS, Exercise 10.1.6, page 228. Use the multivariate CLT to show that  $\mathbb{S}_n \rightarrow_{f.d.} \mathbb{S}$  where  $\mathbb{S}_n$  is the partial sum process of i.i.d.  $(0, 1)$  rv's and  $\mathbb{S}$  is a standard Brownian motion.
- Goldstein's probabilistic proof of the Lindeberg-Feller CLT relies on the following lemma, which is a kind of converse for Slutsky's lemma.  
Let  $\{U_n\}$  and  $\{V_n\}$  be sequences of random variables such that  $U_n$  and  $V_n$  are independent for every  $n$ . Then  $U_n \rightarrow U$  and  $U_n + V_n \rightarrow_d U$  implies that  $V_n \rightarrow_p 0$ . Prove this lemma. (This is Lemma 5.1 in Goldstein (2009).
- Optional bonus problem:** A very famous theorem conjectured by Lévy and proved by Cramér (1936) says that if  $X$  and  $Y$  are independent random variables with  $X + Y = Z$  having a Normal distribution, then both  $X$  and  $Y$  have normal distributions. Find a statement and proof of this theorem. What are the crucial ingredients of the proof?