Statistics 523, Midterm Exam

Wellner; 5/1/2020

Instructons:

A. This is a "take-home' exam. You may use your notes or other books. If your solution to a particular problem is based on some known reference, please acknowledge the source of your solution.

B. You must do this exam completely on your own, with absolutely **no discussion** with other students.

C. Please "hand in" your exam by sending it to me by e-mail before **5 PM on Friday, 8 May**.

1. (30 points) Suppose that ϕ is the characteristic function (of some random variable X). Show that the real part of ϕ (or $Re\phi$) is also a characteristic function.

2. (48 points)
A. Suppose that X has $E(X) = 0$ and $Var(X) = \sigma^2 < \infty$, and write X^* for a random variable having the $X-$zero bias distribution satisfying $\sigma^2 Ef'(X^*) \; = \; E[Xf(X)]$.
(i) Show that for any real number $a \neq 0$ we have $(aX)^* = aX^*$.
(ii) Show that if $|X| \le C$ for some constant C then $|X^*| \le C$.
B. Now suppose that X is a non-negative random variable with mean $\mu = E(X)$, and write X^s for a random variable having the X-size bias distribution (function) F^s satisfying $E[Xf(X)] = \mu Ef(X^s)$ for all functions f for which $E[Xf(X)]$ exists.
(i) Show that for any real number $a > 0$ we have $(aX)^s = aX^s$.
(ii) Show that if $0 \le X \le C$ for some constant C then $0 \le X^s \le C$.

3. (36 points) If X is a random variable with $E(X) = 0$ and $Var(X) = \sigma^2 < \infty$, then the density f^* of the $X-$zero biased distribution F^* exists and is given by (2.54), page 27, of C-G-S (2011). (They call it p^*.)
(i) Show that if the distribution function F has density f, then $f^* \equiv p^*$ is unimodal, with mode at 0.
(ii) What can you say if F is arbitrary (with mean 0 and finite variance)?

4. (36 points) Suppose that $K(t) \equiv E\{X(1_{[0\le t\le X]} - 1_{[X\le t<0]})\}$ where X is a random variable with $E(X) = 0$. Show that $\int_{\mathbb{R}} K(t)dt = E(X^2)$ and $\int_{\mathbb{R}} |t|K(t)dt = 2^{-1}E|X|^3$.

5. (30 points) Suppose that g and h are non-decreasing functions from $\mathbb{R}$ to $\mathbb{R}$, and let X be a random variable satisfying $Eg^2(X) < \infty$ and $Eh^2(X) < \infty$. Show that $Cov[g(X), h(X)] \ge 0$. **Hint:** Let Y be an independent copy of X and consider

$$E(g(Y) - g(X))(h(Y) - h(X)).$$

Do either problem 6 or problem 7

6. (36 points). Suppose that you are given the law of the iterated logarithm for Brownian motion $\mathbb{S}$ at ∞:

$$\limsup_{t\to\infty} \frac{\mathbb{S}(t)}{\sqrt{2t\log\log t}} = 1 \qquad a.s. \tag{1}$$

(a) Prove the *time reversal* property of Brownian motion: if $\mathbb{S}$ is standard Brownian motion, then the process $\tilde{\mathbb{S}}(t) \equiv t\mathbb{S}(1/t)$ is also standard Brownian motion.
(b) Use (a) together with (1) to prove the LIL for Brownian motion at 0:

$$\limsup_{t\to 0} \frac{\mathbb{S}(t)}{\sqrt{2t\log\log(1/t)}} = 1 \qquad a.s. \tag{2}$$

7. (36 points). Suppose that $\mathbb{S}$ is standard Brownian motion on $[0, \infty)$, and define $\mathbb{Y}(t) \equiv e^{-t}\mathbb{S}(e^{2t})$ for $t \in \mathbb{R}$.
(a) Compute $E(\mathbb{Y}(t))$ and $Var(\mathbb{Y}(t))$ for $t \in \mathbb{R}$.
(b) Compute $Cov(\mathbb{Y}(s), \mathbb{Y}(t))$ for $s, t \in \mathbb{R}$.

(c) What is the joint distribution of $(\mathbb{Y}(s), \mathbb{Y}(t))$?
(d) Show that $\mathbb{Y}$ is a stationary process.
(e) Is there a connection between $\mathbb{Y}$ and a Brownian bridge process $\mathbb{U}$ (perhaps divided by $\sqrt{t(1-t)}$)?

8. **Bonus Problem:** (40 points)
Find necessary and sufficient conditions for the CLT in sampling without replacement. One way of proceeding might be to specialize the hypotheses of the theorem of Hájek (1961) to the special case of sampling without replacement in which $a_{i,j} = b_i c_j$ for arbitrary (distinct) numbers $\{c_1, \ldots, c_N\}$ and $b_1 = b_2 = \ldots = b_n = 1$, $b_{n+1} = \cdots = b_N = 0$ where $1 \le n \le N$. Then with $\pi = (\pi_1, \ldots, \pi_N)$ a random permutation of $(1, \ldots, N)$, the sum

$$Y = \sum_{i=1}^{N} b_i c_{\pi(i)}$$

$\stackrel{d}{=}$ the sum of the numbers drawn in sampling n balls without replacement from an urn containing N balls numbered by the c_j's .