Statistics 523, Problem Set 6, Solution

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1. Find a random variable Y with distribution function F having $EY^2 = \infty$ but with $F \in \mathcal{D}(Normal)$.

   Solution: One solution to this is given as follows:
   let $f(x) = cx^{-3}(\log x)^2 1_{[e,\infty)}(x)$. A family of symmetric examples of this type is given by

   $$f(x) = c|x|^{-3}(\log|x|)^r 1_{[e,\infty)}(|x|)$$

   with $r \ge -1$. Now for $r > -1$ we have

   $$\begin{aligned} U(x) &= 2\int_e^x y^2 f_r(y)dy = 2c\int_e^x y^{-1}(\log y)^r d \\ &= 2c\int_1^{\log x} v^r dv = \frac{2c}{r+1}\{(\log x)^{r+1} - 1\} \\ &\sim \frac{2c}{r+1}(\log x)^{r+1} \quad \text{as} \quad x \to \infty. \end{aligned}$$

   Thus with $A_n = dn^{1/2}(\log n)^{(r+1)/2}$ we have

   $$\begin{aligned} \frac{nU_n(A_n)}{A_n^2} &\sim \frac{2cn(\log A_n)^{r+1}}{A_n^2} = \frac{2cn(\log(dn^{1/2}(\log n)^{r/2}))^{r+1}}{d^2 n(\log n)^{r+1}} \\ &\to \frac{2c}{2^{r+1}d^2} = 1 \end{aligned}$$

   if $d = \sqrt{c/2^r}$. Thus we conclude that for $r > -1$

   $$\frac{S_n - n\mu}{\sqrt{(c/2^r)n(\log n)^{r+1}}} \to_d Z \sim N(0,1).$$

   When $r = -1$ we find that

   $$U(x) = 2c\int_1^{\log x} v^{-1}dv = 2c\log\log x,$$

and then with $A_n = dn^{1/2}(\log\log n)^{1/2}$ we have

$$\begin{aligned} \frac{nU(A_n)}{A_n^2} &= \frac{2cn\log\log(dn^{1/2}(\log\log n)^{1/2}}{d^2 n\log\log n} \\ &\to 1 \end{aligned}$$

if $d = \sqrt{2c}$. Therefore in this case it follows that

$$\frac{S_n}{\sqrt{2cn\log\log n}} \to_d Z \sim N(0,1).$$

Note that if $r < -1$, then $r+1 < 0$ and

$$U(x) = \frac{2c}{-(r+1)}\left(1 - (\log x)^{r+1}\right) \to \frac{2c}{-(r+1)} \equiv \sigma_r^2 < \infty,$$

so $E_r X^2 = Var_r(X) = \sigma_r^2 < \infty$, and $(S_n - n\mu)/\sqrt{n} \to_d N(0, \sigma_r^2)$.

2. Use a reflection principle to show that for $0 \le y \le x$

$$P(\sup_{0\le s\le t} \mathbb{S}(s) \ge x,\ \mathbb{S}(t) \le y) = P(\mathbb{S}(t) \ge 2x - y)\,,$$

and use this to show that the joint density of $M^+ \equiv \sup_{0\le s\le t} \mathbb{S}(s)$ and $\mathbb{S}(t)$ is given by

$$f(x,y) = \sqrt{\frac{2}{\pi t^3}}(2x-y)\exp\left(-\frac{(2x-y)^2}{2t}\right) \qquad \text{for } \ 0 \le y \le x\,.$$

Check to make sure that this gives the correct marginal densities.

Solution: Note that by reflection

$$\begin{aligned} P(\sup_{0\le s\le t} \mathbb{S}(s) \ge x,\ \mathbb{S}(t) \le y) &= P(\tau_x \le t, \mathbb{S}(\tau_x + t - \tau_x) - \mathbb{S}(\tau_x) \le y - x) \\ &= P(\tau_x \le t, \mathbb{S}(\tau_x + t - \tau_x) - \mathbb{S}(\tau_x) \ge x - y) \\ &= P(\tau_x \le t, \mathbb{S}(t) \ge 2x - y) \\ &= P(\mathbb{S}(t) \ge 2x - y) \\ &= 1 - \Phi\left(\frac{2x-y}{\sqrt{t}}\right) \end{aligned}$$

since $[\mathbb{S}(t) \geq 2x - y] \subset [\tau_x \leq t]$. Thus we compute the joint density f of $M^+ \equiv \sup_{0\leq s\leq t} \mathbb{S}(s), \mathbb{S}(t)$ as

$$\begin{aligned} f(x,y) &= -\frac{\partial^2}{\partial x \partial y} P(\sup_{0\leq s\leq t} \mathbb{S}(s) \geq x,\ \mathbb{S}(t) \leq y) \\ &= \frac{\partial}{\partial y}\phi\left(\frac{2x-y}{\sqrt{t}}\right)\frac{2}{\sqrt{t}} \\ &= \phi'\left(\frac{2x-y}{\sqrt{t}}\right)\frac{-1}{\sqrt{t}}\frac{2}{\sqrt{t}} \\ &= \frac{2x-y}{\sqrt{t}}\phi\left(\frac{2x-y}{\sqrt{t}}\right)\frac{2}{t} \\ &= \sqrt{\frac{2}{\pi t^3}}(2x-y)\exp\left(-\frac{(2x-y)^2}{2t}\right) \qquad \text{for } \ 0 \leq y \leq x \end{aligned}$$

where we used $\phi'(x) = -x\phi(x)$ in the next to last equality.

3. PfS Course Notes, Exercise 12.9.2, page 332:
Let $V_n(r) \equiv \sum_{k=1}^n |\mathbb{S}(t_{nk}) - \mathbb{S}(t_{n,k-1})|^r$ and $\|\mathcal{P}_n\| \equiv \sup_{1\leq k\leq n} |t_{nk} - t_{n,k-1}|$.
(α) Let $Z \sim N(0,1)$. Let $r > 0$. Show that:
  (a) $C_r \equiv E|Z|^r = 2^{r/2}\Gamma((r+1)/2)/\sqrt{\pi}$.
  (b) $E|\mathbb{S}(t_{n,k-1}, t_{n,k}]|^r = C_r|t_{n,k} - t_{n,k-1}|^{r/2}$,
  $Var(|\mathbb{S}(t_{n,k-1}, t_{n,k}]|^r) = (C_{2r} - C_r^2)|t_{n,k} - t_{n,k-1}|^r$.
(β) Now show that $EV_n(2) = 1$ and $Var(V_n(2)) \leq (C_{2r} - C_r^2)\|\mathcal{P}_n\|$ and hence

$$\sum_{n=1}^\infty P(|V_n(2) - 1| \geq \epsilon) \leq \epsilon^{-2}(C_{2r} - C_r^2)\sum_{n=1}^\infty \|\mathcal{P}_n\|.$$

(γ) Finally prove that $V_n(2) \to_{a.s.} 1$ if $\sum_{n=1}^\infty \|\mathcal{P}_n\| < \infty$.

Solution: (α) First, if $Z \sim N(0,1)$ and $r > 0$

$$\begin{aligned} E|Z|^r &= 2\int_0^\infty z^r (2\pi)^{-1/2}\exp(-z^2/2)dz \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty (2v)^{(r-1)/2}e^{-v}dv \\ &= \sqrt{\frac{2}{\pi}}2^{(r-1)/2}\Gamma((r+1)/2) = 2^{r/2}\Gamma((r+1)/2)/\sqrt{\pi} \equiv C_r. \end{aligned}$$

Thus $C_{2k} = 1 \cdot 3 \cdots (2k-1)$. Then note that $\mathbb{S}(t_{n,k}) - \mathbb{S}(t_{n,k-1}) \sim N(0, t_{n,k} - t_{n,k-1})$ and hence

$$E|\mathbb{S}(t_{n,k-1}, t_{n,k}]|^r = |t_{n,k} - t_{n,k-1}|^{r/2} E|Z|^r = C_r |t_{n,k} - t_{n,k-1}|^{r/2}.$$

and

$$\begin{aligned} Var(\mathbb{S}(t_{n,k-1}, t_{n,k}]^2) &= E|\mathbb{S}(t_{n,k-1}, t_{n,k}]|^4 - \{E|\mathbb{S}(t_{n,k-1}, t_{n,k}]|^2\}^2 \\ &= (C_4 - C_2^2)|t_{n,k} - t_{n,k-1}|^2 \end{aligned}$$

as claimed (and with $r = 2$).
(β) Now it follows from (α) that

$$EV_n(2) = C_2 \sum_{k=1}^{n} (t_{n,k} - t_{n,k-1}) = 1 \cdot \sum_{k=1}^{n} (t_{n,k} - t_{n,k-1}) = 1$$

since $C_2 = 1$, and using the independence of the increments of Brownian motion,

$$\begin{aligned} Var(V_n(2)) &= (C_4 - C_2^2) \sum_{k=1}^{n} (t_{n,k} - t_{n,k-1})^2 \\ &\le (3-1)\|\mathcal{P}_n\| \sum_{k=1}^{n} (t_{n,k} - t_{n,k-1}) = 2\|\mathcal{P}_n\| \end{aligned}$$

as claimed (but with $r = 2$). Thus by Markov's inequality,

$$\sum_{n=1}^{\infty} P(|V_n(2) - 1| \ge \epsilon) \le 2\epsilon^{-2} \sum_{n=1}^{\infty} \|\mathcal{P}_n\|.$$

(γ) This last display together with the Borel-Cantelli lemma shows that $P(|V_n(2) - 1| \ge \epsilon \text{ i.o.}) = 0$ for every $\epsilon > 0$, and hence $V_n(2) \to_{a.s.} 1$ if $\sum_{n=1}^{\infty} \|\mathcal{P}_n\| < \infty$.