Statistics 523, Problem Set 5 Solutions

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1. Suppose that $a_n \nearrow$ with $a_1 = 1$ and suppose that $a_{mk} = a_m a_k$ for all $k, m \geq 1$. Show that $a_n = n^{1/\alpha}$ for some $\alpha \geq 0$.

   Solution: Fix $k \geq 1$. Then for $j \geq 1$,

   $$a_{k^j} \;=\; a_{k \cdot k^{j-1}} = a_k a_{k^{j-1}} = \cdots = (a_k)^j .$$

   For $n > k$, let j satisfy $k^j \leq n \leq k^{j+1}$. Then

   $$\begin{aligned} & a_{k^j} \leq a_n \leq a_{k^{j+1}}, \quad \text{or} \\ & \log a_{k^j} \leq \log a_n \leq \log a_{k^{j+1}}, \quad \text{or, by (1)} \\ & j \log a_k \leq \log a_n \leq (j+1) \log a_k . \end{aligned}$$

   Now divide by $j \log k$ to see that

   $$\frac{\log a_k}{\log k} \leq \frac{\log a_n}{\log n} \cdot \frac{\log n}{j \log k} \leq \frac{j+1}{j} \cdot \frac{\log a_k}{\log k}. \tag{1}$$

   Now when $n \to \infty$ we also have $j \to \infty$ and $\log n/(j \log k) \to 1$. Combined with (1) this yields

   $$\frac{\log a_k}{\log k} = \lim_{n \to \infty} \frac{\log a_n}{\log n} = \text{some } c > 0.$$

   Thus we conclude that $a_k = k^c$. For some c.

2. Assume that Y with distribution function G is stable with characteristic exponent α. Show that $E|Y|^r < \infty$ for all $0 < r < \alpha$. [Hint: use the inequalities of PfS Section 8.3 to show that $nP(|X| > a_n x)$ is bounded in n, where $a_n \equiv n^{1/\alpha}$. Then bound the appropriate integral.

   Solution: Note that $G \in \mathcal{D}_N(G)$. Thus it follows from the characterization of the normal domain of attraction Theorem 4.1(c) stated in class and Exercise 11.4.1 that for some constant $c < \infty$ and $0 \leq p \leq 1$

   $$x^\alpha P(Y > x) \to cp \;\text{ and }\; x^\alpha P(Y < -x) \to c(1-p)$$

   as $x \to \infty$. Thus we also have

   $$x^\alpha P(|Y| > x) \to c \;\text{ as }\; x \to \infty,$$

and hence $x^\alpha P(|Y| > x) \le 2c$ for $x \ge$ some x_0. Thus we have, for $0 < r < \alpha$,

$$\begin{aligned}
E|Y|^r &= \int_0^\infty r x^{r-1} P(|Y| \ge x) dx \\
&\le \int_0^{x_0} r x^{r-1} dx + \int_{x_0}^\infty r x^{r-1} x^{-\alpha} x^\alpha P(|Y| \ge x) dx \\
&\le x_0^r + \int_{x_0}^\infty r x^{r-1-\alpha} 2c dx \\
&\le x_0^r + \frac{2rc}{\alpha - r} < \infty.
\end{aligned}$$

3. (a) Let Y be a stable random variable with $\theta = 1$ and $0 < \alpha < 1$. Show that $P(Y \ge 0) = 1$.
(b) Let Y be as in (a). By the conclusion of (a) the Laplace transform of Y, $\psi(\lambda) = E\exp(-\lambda Y)$ is well-defined. Show that $Y_1 + \cdots + Y_k \overset{d}{=} a_k Y + b_k$ holds with $b_k = 0$ (and $Y_1, \ldots, Y_k$ i.i.d. as Y).
(c) Show that $\psi(\lambda)^n = \psi(n^{1/\alpha}\lambda)$ and hence that $\psi(\lambda) = \exp(-c\lambda^\alpha)$ for some $c > 0$.

Solution: (a) Suppose that $P(Y \ge 0) = 1$ and that Y is stable with $\alpha \in (0, 1)$ and $\theta = 1$. Then from our development of the characteristic function of an infinitely divisible random variable, the Lévy- Khintchine representation of the characteristic function of Y is of the form

$$\begin{aligned}
\phi_Y(t) &= \exp\left(itc + m_1 \int_0^\infty \left(e^{itx} - 1 - it\frac{x}{1+x^2} \right) \frac{1}{x^{\alpha+1}} dx \right) \\
&= \exp(it\mu - d|t|^\alpha(1 - i\text{sign}(t)C_\alpha))
\end{aligned}$$

where $d = \alpha^{-1}\Gamma(1-\alpha)m_1 \cos(\pi\alpha/2)$ and $C_\alpha = \tan(\pi\alpha/2)$. Since Y is stable we have

$$\phi_Y(t)^k = \phi_Y(a_k t) e^{itb_k}$$

for all $k \ge 1$, and since $a_k = k^{1/\alpha}$ this yields

$$\exp(itk\mu - dk|t|^\alpha(1 - \text{sign}(t)C_\alpha)) = \exp(ik^{1/\alpha}t\mu - dk|t|^{1/\alpha}(1 - \text{sign}(k^{1/\alpha}t)C_\alpha))e^{itb_k},$$

for all t and all k, so we see that $k\mu = k^{1/\alpha}\mu + b_k$ or $-b_k = (k^{1/\alpha} - k)\mu$ and hence $-b_k/a_k = (1 - k^{1-1/\alpha})\mu \to \mu$ as $k \to \infty$ since $1 - 1/\alpha < 0$ for $0 < \alpha < 1$. Thus it follows from stability of Y and $Y \in \mathcal{D}_N(G_\alpha)$ that on the one hand

$$\phi_{(S_k - b_k)/a_k}(t) = \phi_Y(t) = \exp(it\mu - d|t|^\alpha(1 - \text{sign}(t)C_\alpha)),$$

while on the other hand

$$\phi_{(S_k-b_k)/a_k}(t) = Ee^{itS_k/a_k}\exp(-itb_k/a_k)$$

where $-b_k/a_k \to \mu$. It follows that

$$E\exp(itS_k/a_k) \to \exp(-d|t|^\alpha|t|^\alpha(1-\operatorname{sign}(t)C_\alpha)).$$

Since we started with $P(Y \geq 0) = 1$ it follows that $P(S_k \geq 0) = 1$ and hence the limiting distribution of $S_k/k^{1/\alpha}$, namely the distribution with chf $\exp(-d|t|^\alpha|t|^\alpha(1-\operatorname{sign}(t)\tan(\pi\alpha/2)))$, corresponds to the chf of a random variable which is non-negative with probability 1. Since this characteristic function is that of an $\alpha-$stable random variable $\tilde{Y}$ with $\mu = 0$ and $\theta = 1$, by uniqueness of characteristic functions we conclude that $P(\tilde{Y} \geq 0) = 1$.

(b) Since $Y \geq 0$ a.s. we have $\exp(-\lambda Y) \leq 1$ a.s., and hence $\psi(\lambda) = E\exp(-\lambda Y)$ is well-defined. From the considerations in the proof of (a) we have $b_k = -(k^{1/\alpha} - k)\mu = 0$ for all k.

(c) Since $a_k Y \overset{d}{=} Y_1 + \cdots + Y_k$ for every $k \geq 1$ with $a_k = k^{1/\alpha}$, it follows that

$$\psi(\lambda)^k = \psi(k^{1/\alpha}\lambda) \qquad \text{for all} \quad \lambda \geq 0, \quad k \geq 1.$$

Now the proof proceeds much as our proof for the identification of the measures $M^\pm$ in the stable case. Set $\lambda = (n/k)^{1/\alpha}$: then it follows that

$$k\log\psi((n/k)^{1/\alpha}) = \log\psi(k^{1/\alpha}(n/k)^{1/\alpha}) = \log\psi(n^{1/\alpha}) \ \text{ for all } \quad n,k \geq 1.$$

In particular, with $k = n$ this yields $n\log\psi(1) = \log\psi(n^{1/\alpha})$, and then substitution of this in the last display yields

$$k\log(\psi((n/k)^{1/\alpha})) = n\log\psi(1), \quad \text{or} \quad \log(\psi((n/k)^{1/\alpha})) = (n/k)\log\psi(1).$$

for all $n,k \geq 1$. Thus for all λ in the dense set $\{(n/k)^{1/\alpha}\}$ we have shown that

$$\log\psi(\lambda) = \lambda^\alpha \log\psi(1).$$

Since ψ is monotone decreasing, this implies that

$$\psi(\lambda) = \exp(-c\lambda^\alpha) \tag{2}$$

where $c \equiv -\log\psi(1) > 0$. Note that this agrees with the characteristic function as identified in (a): taking $-\lambda = it$ we have $\lambda = -ti = te^{-i\pi/2}$ for $t > 0$, and hence $\lambda^\alpha = t^\alpha e^{-i\pi\alpha/2}$. It follows that

$$\begin{aligned} -c\lambda^\alpha &= -ct^\alpha e^{-i\pi\alpha/2} = -t^\alpha\{\cos(\pi\alpha/2) - i\sin(\pi\alpha/2)\} \\ &= -c\cos(\pi\alpha/2)t^\alpha\{1 - i\tan(\pi\alpha/2)\} \end{aligned}$$

where $c \geq 0$. This is exactly the form of the $\alpha-$stable characteristic function with $\mu = 0$ and $\theta = 1$ as in (a).

Remark: See *Stable Non-Gaussian Random Processes* by Samorodnitsky and Taqqu, pages 13 - 20 for more about completely asymmetric stable distributions and their properties.

Here is a second (more explicit) solution of part (a). (a) Suppose that $0 < \alpha < 1$. Let X have $P(X > x) = (1+x)^{-\alpha}$ for all $x \geq 0$ (and necessarily $P(X > x) = 1$ for $x < 0$). Then

$$x^{\alpha} P(X > x) = \left(\frac{x}{1+x}\right)^{\alpha} \to 1 \quad \text{as} \quad x \to \infty,$$

and $P(X > x)/P(|X| > x) = 1 \to 1 \equiv p$. Thus $\mathcal{L}(X) \in \mathcal{D}_N(G)$ where G is stable with exponent $\alpha \in (0,1)$. We compute the limiting characteristic function of $X_1 + \ldots + X_n$ directly along the same lines of argument used for the example in class (from Durrett). Now

$$1 - Ee^{itX} = \int_0^{\infty} (1 - e^{itx}) \alpha (1+x)^{-(\alpha+1)} dx.$$

Thus, for $t > 0$,

$$\begin{aligned}
Re(1 - Ee^{itX}) &= \int_0^{\infty} (1 - \cos(tx)) \alpha (1+x)^{-(1+\alpha)} dx \\
&= \int_1^{\infty} (1 - \cos(t(y-1))) \alpha y^{-(\alpha+1)} dy \\
&= \alpha t^{\alpha} \int_t^{\infty} (1 - \cos(u - t)) u^{-(\alpha+1)} du \\
&= \alpha t^{\alpha} \int_t^{\infty} \frac{1 - \cos(u-t)}{(u-t)^{\alpha+1}} \cdot \frac{(u-t)^{\alpha+1}}{u^{\alpha+1}} du \\
&\sim \alpha t^{\alpha} \int_0^{\infty} \frac{1 - \cos(u)}{u^{\alpha+1}} du \quad \text{as} \quad t \to 0 \\
&= \alpha C_{\alpha} t^{\alpha}.
\end{aligned}$$

The integral appearing in the limit in the last display is the same as the integral that appeared in class, and, in fact,

$$C_{\alpha} = -\Gamma(-\alpha) \cos(\pi\alpha/2).$$

Similarly, for $t > 0$,

$$Im(1 - Ee^{itX}) \sim -\alpha D_{\alpha} t^{\alpha},$$

and we conclude that for $t > 0$

$$1 - Ee^{itX} \sim \alpha C_\alpha t^\alpha \left\{1 - i\text{sign}(t) D_\alpha / C_\alpha\right\}.$$

Thus it follows that with $X_1, \ldots, X_n$ i.i.d. as X, $S_n \equiv X_1 + \cdots X_n$, and $t > 0$,

$$\begin{aligned} E\exp(itn^{-1/\alpha} S_n) &= (1 - (1 - E\exp(itn^{-1/\alpha} X_1))^n \\ &\to \exp\left(-\alpha C_\alpha t^\alpha \{1 - i\text{sign}(t) D_\alpha / C_\alpha\right) \end{aligned}$$

which is the characteristic function of a stable law with $d = 0$, $\theta = 1$, and $c = \alpha C_\alpha$ and (necessarily) $D_\alpha / C_\alpha = \tan(\pi\alpha/2)$. Thus $n^{-1/\alpha} S_n \to Y \sim S_\alpha(c, 1, 0)$. Since $P(X \geq 0) = 1$, it follows that $P(n^{-1/\alpha} S_n \geq 0) = 1$ and hence $P(Y \geq 0) = 1$.

4. Let $X_1, X_2, \ldots$ be i.i.d. with a density function f that is symmetric about 0 and continuous and positive at 0. Show that

$$\frac{1}{n}\left(\frac{1}{X_1} + \cdots + \frac{1}{X_n}\right) \to_d Y$$

where Y has a Cauchy distribution. Hint: See Durrett, PTE, Example 3.71, page 165.

Solution: Let $Y_i \sim 1/X_i$ for $i \geq 1$. Then

$$P(1/X_1 > x) = P(0 < X < 1/x) = \int_0^{1/x} f(y)dy \sim x^{-1} f(0) \quad \text{as} \quad x \to \infty$$

and similarly

$$P(1/X_1 < -x) = P(0 < X < 1/x) \sim x^{-1} f(0) \quad \text{as} \quad x \to \infty$$

so $P(Y > x)/P(|Y| > x) \to 1/2$ while $1 - F(y) \sim y^{-1} f(0)$; here $F = F_Y$ is the distribution function of $Y = 1/X$, while f is the density of X. Thus $Y \sim F$ is in the domain of attraction of a symmetric Cauchy distribution with $\alpha = 1$ and characteristic function of the form $\exp(-d|t|)$ for some $d > 0$, and we conclude that the asserted convergence holding for some Cauchy distribution of the form dC where $C \sim \text{Cauchy}(0, 1)$ and $d > 0$. To identify d we proceed by direct calculation of the chf of Y, much as in the symmetric example considered in class:

$$\begin{aligned} \phi_Y(t) &= Ee^{itY} = Ee^{it/X} = \int_{-\infty}^{\infty} e^{it/x} f(x)dx \\ &= \int_{-\infty}^{\infty} \cos(t/x) f(x)dx \\ &= 2\int_0^\infty \cos(ty) f(1/y) y^{-2} dy \quad \text{by symmetry of} \quad f. \end{aligned}$$

Thus, using $2\int_0^\infty f(1/y)y^{-2}dy = 1$,

$$\begin{aligned} 1 - \phi_Y(t) &= 2\int_0^\infty (1-\cos(ty))f(1/y)y^{-2}dy = 2t\int_0^\infty (1-\cos(w))f(t/w)w^{-2}dw \\ &\sim 2tf(0)\int_0^\infty (1-\cos(w))w^{-2}dw = tf(0)\pi \end{aligned}$$

as $t \searrow 0$ since $\int_0^\infty (1-\cos(w))w^{-2}dw = \pi/2$. Since $\phi_Y(-t) = \phi_Y(t)$ by symmetry of Y this yields

$$1 - \phi_Y(t) \sim \pi f(0)|t| \quad \text{as} \quad t \to 0.$$

Hence with $a_n = n$ and $S_n = \sum_{i=1}^n Y_i = \sum_{i=1}^n X_i^{-1}$,

$$\begin{aligned} \phi_{S_n/n}(t) &= \phi_Y(t/n)^n = \left(1 - \frac{n(1-\phi_Y(t/n))}{n}\right)^n \\ &\to \exp(-\pi f(0)|t|), \end{aligned}$$

and it follows that $S_n/n \to_d \pi f(0)C$ where $C \sim \text{Cauchy}(0,1)$.

5. (i) Suppose that X is symmetric stable with index α and $Y \geq 0$ is an independent stable with index $\beta < 1$, then $XY^{1/\alpha}$ is symmetric stable with index $\alpha\beta$.
   (ii) Suppose that Z_1 and Z_2 are independent $N(0,1)$ random variables. Check that $1/Z_1^2$ has the density of T_1 the first time Brownian motion starting from 0 hits the level 1. Use this to show that Z_1/Z_2 has a Cauchy distribution.

   Solution: (i) If X has a symmetric stable distribution with index $\alpha \in (0,2]$ and $Y \geq 0$ is stable with index $\beta < 1$, then $W \equiv XY^{1/\alpha}$ has characteristic function

$$\begin{aligned} \phi_W(t) &= Ee^{itW} = Ee^{itXY^{1/\alpha}} = E\{E(e^{itXY^{1/\alpha}}|Y)\} \\ &= E\exp(-d|t|^\alpha Y) = \exp(-c(d|t|^\alpha)^\beta) \\ &= \exp(-cd^\beta|t|^{\alpha\beta}) \end{aligned}$$

   by using Problem 3 at the next to last step. Here c and d are positive constants and $0 < \alpha\beta < \alpha \leq 2$.
   (ii) Note that $1/Z_2^2$ has distribution function

$$\begin{aligned} F_{1/Z_2^2}(v) &= P(1/Z_2^2 \leq v) = P(|Z_2| \geq 1/\sqrt{v}) = 2P(Z_2 \geq 1/\sqrt{v}) \\ &= 2(1-\Phi(1/\sqrt{v})), \end{aligned}$$

   and hence $1/Z_2^2$ has density given by

$$f(v) = 2\phi(1/\sqrt{v})(1/2)v^{-3/2} = \frac{1}{\sqrt{2\pi v^3}}\exp(-1/2v)1_{(0,\infty)}(v).$$

Thus $Y \equiv 1/Z_2^2 \overset{d}{=} T_1$ the first hitting time of level 1 for a standard Brownian motion, which is (completely asymmetric) stable with index $\beta = 1/2$. Furthermore, $X \equiv Z_1 \sim N(0,1)$ is symmetric stable with index $\alpha = 2$ Thus we conclude from (i) that $W = XY^{1/2} = Z_1/|Z_1|$ has characteristic function of the form

$$\phi_W(t) = E\exp(itW) = \exp(-\tilde{c}|t|^{2\cdot(1/2)} = \exp(-\tilde{c}|t|).$$

It follows that $W = Z_1/|Z_2| \sim \ \tilde{c}C$ where $C \sim$ standard Cauchy. Since Z_1 and Z_2 are both symmetric, the same is true for $\tilde{W} = Z_1/Z_2$.

For some further interesting (multivariate) results of this type, see Pillai and Meng (2016), *Ann. Statist.* **44**, 2089 - 2097.