Statistics 523, Problem Set 3 Solutions

Wellner; 4/19/2017

1. (a) Give an example of a random variable Y with distribution function F on $\mathbb{R}^+ = [0, \infty)$ for which $EY^r = \infty$ for all $r > 0$.
(b) Does your example in (a) have $Eg(Y) < \infty$ for some measurable function g with $g(y) \to \infty$ as $y \to \infty$?

   Solution: (a) Suppose that F is defined by

$$1 - F(x) = \begin{cases} (\log x)^{-\gamma}, & x \ge e, \\ 1, & x < e, \end{cases}$$

   where $\gamma > 0$. Note that F has density f given by $f(x) = \gamma x^{-1} (\log x)^{-1-\gamma} 1_{[e,\infty)}(x)$. Then if $Y \sim F$,

$$\begin{aligned} EY^r &= \int_0^\infty r x^{r-1}(1 - F(x))dx = \int_0^e r x^{r-1} dx + \int_e^\infty r x^{r-1} (\log x)^{-\gamma} dx \\ &= \infty \quad \text{for all} \quad r > 0 \end{aligned}$$

   since $\lim_{x\to\infty} x^r/(\log x)^\gamma = \infty$ for all $r, \gamma > 0$.
(b) Consider $g(x) = (\log x)^\delta$. Then $g(x) \to \infty$ if $\delta > 0$. Moreover,

$$\begin{aligned} Eg(Y) &= \int_e^\infty (\log x)^\delta \frac{\gamma}{x(\log x)^{\gamma+1}} dx \\ &= \gamma \int_1^\infty v^{-(1+\gamma-\delta)} dv = \frac{\gamma}{\gamma - \delta} < \infty \end{aligned}$$

   if $0 < \delta < \gamma$.

2. A very famous theorem conjectured by Lévy and proved by Cramér (1936) says that if X and Y are independent random variables with $X + Y = Z$ having a Normal distribution, then both X and Y have normal distributions. Find a statement and proof of this theorem. What are the crucial ingredients of the proof?

   Solution: One easy source for this is Feller's "An Introduction to Probability Theory and Its Applications, Vol II", page 525. Another source is Pollard's "A User's Guide to Measure Theoretic Probability", section 8.8, pages 205-206. Pollard gives a proof with references to the relevant results from complex analysis as given in Rudin (1974), "Real and Complex Analysis". Other sources include W. Bryc (1995),"The Normal Distribution: Characterizations with Applications", and Kagan, Linnik, and Rao (1973), "Characterization problems in mathematical statistics".

The proof depends on the following interesting lemma concerning characteristic functions:

Lemma: Let F be a probability distribution such that

$$g(\eta) = \int_{-\infty}^{\infty} \exp(\eta^2 x^2) dF(x) < \infty$$

for some $\eta > 0$. Then the characteristic function φ of F is an entire function defined for all complex z. If $\varphi(z) \neq 0$ for all complex z, then F is normal.

An equivalent modern statement of the exponential integrability hypothesis would be to say that $X \sim F$ has a finite ψ_2- Orlicz norm $\|X\|_{\psi_2}$ where $\psi_2(x) = \exp(x^2) - 1$.

Another statement of the Theorem (from Chow and Teicher (1978), page 282) is as follows:

Theorem: (Cramér - Lévy) The family of normal distributions is factor closed.

It turns out that the families of Poisson distributions and binomial distributions are also factor closed (to within translations); see Chow and Teicher (1978), Theorem 4, page 283.

3. Stein's method for convergence in distribution to the Poisson distribution depends on the following characterization: $X \sim \text{Poisson}(\lambda)$ if and only if

$$E[Xf(X)] = \lambda E[f(X+1)]$$

for all functions f for which the expectations exist. Show that if $X \sim \text{Poisson}(\lambda)$ then the identity in the display holds for any bounded function f.

Proof. Suppose that $X \sim \text{Poisson}(\lambda)$ and let $f : \mathbb{N} \to \mathbb{R}$ be bounded. Then

$$\begin{aligned} E[Xf(X)] &= \sum_{k=0}^{\infty} kf(k)e^{-\lambda}\frac{\lambda^k}{k!} \\ &= \sum_{k=1}^{\infty} f(k)e^{-\lambda}\frac{\lambda^k}{(k-1)!} \\ &= \sum_{m=0}^{\infty} f(m+1)e^{-\lambda}\frac{\lambda^{m+1}}{m!} = \lambda\sum_{m=0}^{\infty} e^{-\lambda}\frac{\lambda^m}{m!} \\ &= \lambda Ef(X+1). \end{aligned}$$

The reverse argument goes as follows: Suppose that for any $A \subset \mathbb{N}$ we can construct a function $g_{A,\lambda} : \mathbb{N} \to \mathbb{R}$ satisfying

$$\lambda g(k+1) - kg(k) = 1_A(k) - \text{Poiss}_\lambda(A) \tag{1}$$

for all $k \geq 0$. Then if W takes values in $\mathbb{N}$ we have

$$\lambda E\{\lambda g(W+1) - Wg(W)\} = P(W \in A) - \text{Poiss}_\lambda(A)$$

If the left side is zero, then we conclude that W has a Poisson(λ) distribution. The solution of (1) can be found recursively, starting from $k = 0$ and working up.

4. Goldstein's probabilistic proof of the Lindeberg-Feller CLT relies on the following lemma, which is a kind of converse for Slutsky's lemma.
   Let $\{U_n\}$ and $\{V_n\}$ be sequences of random variables such that U_n and V_n are independent for every n. Then $U_n \to U$ and $U_n + V_n \to_d U$ implies that $V_n \to_p 0$. Prove this lemma. (This is Lemma 5.1 in Goldstein (2009).)

   Solution: As I noted in class on 17 April, Independence of U_n and V_n for each n together with convergence in distribution of both U_n and $U_n + V_n$ to U yields

   $$\phi_U(t) = Ee^{itU} \leftarrow Ee^{it(U_n+V_n)} = Ee^{itU_n} \cdot Ee^{itV_n} \to \phi_U(t) \cdot \lim_{n\to\infty} Ee^{itV_n},$$

   and hence $\phi_U(t) \lim_{n\to\infty} Ee^{itV_n} = \phi_U(t)$ for all $t \in \mathbb{R}$.

   Two of you turned this into a proof by arguing as follows: Since ϕ_U is a characteristic function (of a proper random variable), there is a neighborhood of 0, say $|t| < \delta$, such that $\phi_U(t) \neq 0$ for all $|t| < \delta$; this follows from $\phi_U(0) = 1$ and the continuity of ϕ_U. This leads to the conclusion that the limit $\phi_V(t) = \lim_{n\to\infty} Ee^{itV_n} = 1$ for $|t| < \delta$ for some (perhaps small) $\delta > 0$. But this implies that $E(V) = 0$ and $E|V|^2 = 0$ by Durrett (2010), exercise 3.3.19. This implies $V = 0$ with probability 1, and hence $\phi_V(t) = 1$ for all $t \in \mathbb{R}$. Thus $V_n \to_d 0$ and this implies that $V_n \to_p 0$.

   Alternatively, Use Exercise 3.3.20, Durrett (2010): $V_n \to_d 0$ if and only if $\phi_{V_n}(t) \to 1$ for $|t| < \delta$ for some $\delta > 0$.

   Of course the point of Goldstein's proof is to (completely!) avoid the use of characteristic functions.

   Here is the statement and solution of Durrett's exercise. If $\lim_{t\searrow 0} t^{-2}(\phi(t) - 1) = c > -\infty$, then $E(X) = 0$ and $E|X|^2 = -2c < \infty$. In particular, if $\phi(t) = 1 + o(t^2)$, then $\phi(t) \equiv 1$.
   Solution: $E|X|^2 < \infty$ follows from Theorem 3.3.9, Durrett (2010). By comparison with $\phi(t) = 1 + it\mu - (1/2)t^2\sigma^2 + o(t^2)$ (Theorem 3.3.8, Durrett (2010)), it follows that $\mu = 0$ and $\sigma^2 = -2c$. If $\phi(t) = 1 + o(t^2)$, then $c = 0$ and $X \equiv 0$.