

Statistics 523, Problem Set 2 Solutions

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1. PfS Course Notes, Exercise 10.2.1, page 236. (Characterization of “uan”) Suppose that $\{X_{n,k} : 1 \leq k \leq n\}$ is a row-independent triangular array with $E(X_{n,k}) = 0$, $E(X_{n,k}^2) \equiv \sigma_{n,k}^2$, normalized so that $\sigma_n^2 \equiv \sum_{k=1}^n \sigma_{n,k}^2 = 1$. Show that the following are equivalent:
- (a) $|X_{n,k}|$'s are uan; that is, $\max_{1 \leq k \leq n} P(|X_{n,k}| \geq \epsilon) \rightarrow 0$ for all $\epsilon > 0$.
 - (b) $\max_{1 \leq k \leq n} |\phi_{nk}(t) - 1| \rightarrow 0$ uniformly on every finite interval of t 's.
 - (c) $\max_{1 \leq k \leq n} E(X_{n,k}^2 \wedge 1) = \max_{1 \leq k \leq n} \int (x^2 \wedge 1) dF_{nk}(x) \rightarrow 0$.

Solution: Suppose that (a) holds. We first show that something more general than (c) holds. Fix $\epsilon > 0$, $r > 0$ and $\tau > \epsilon$. Then we can write

$$\begin{aligned} E|X_{nk}|^r 1_{[|X_{nk}| \leq \tau]} &= E|X_{nk}|^r 1_{[|X_{nk}| \leq \epsilon]} + E|X_{nk}|^r 1_{[\epsilon < |X_{nk}| \leq \tau]} \\ &\leq \epsilon^r + \tau^r P(|X_{nk}| > \epsilon). \end{aligned}$$

Thus

$$\begin{aligned} \max_{k \leq n} E|X_{nk}|^r 1_{[|X_{nk}| \leq \tau]} &\leq \epsilon^r + \tau^r \max_{k \leq n} P(|X_{nk}| > \epsilon) \\ &\rightarrow \epsilon^r \end{aligned}$$

for every $0 < \epsilon < \tau$. But since ϵ is arbitrary, this yields

$$\max_{k \leq n} E|X_{nk}|^r 1_{[|X_{nk}| \leq \tau]} \rightarrow 0$$

Hence for every $r > 0$ and $\tau > 0$

$$\max_{k \leq n} E\{|X_{nk}|^r \wedge \tau^r\} = \max_{k \leq n} \{\tau^r P(|X_{nk}| > \tau) + E\{|X_{nk}|^r 1_{[|X_{nk}| \leq \tau]}\}\} \rightarrow 0$$

as $n \rightarrow \infty$. In particular with $r = 2$ and $\tau = 1$ we have $\max_{k \leq n} E\{|X_{nk}|^2 \wedge 1\} \rightarrow 0$.

Conversely, suppose (c) holds and let $0 < \epsilon < \tau$. Then

$$\begin{aligned} P(|X_{nk}| > \epsilon) &= P(|X_{nk}| > \epsilon, |X_{nk}| > \tau) + P(|X_{nk}| > \epsilon, |X_{nk}| \leq \tau) \\ &\leq P(|X_{nk}| > \tau) + \epsilon^{-2} E\{X_{nk}^2 1_{[|X_{nk}| \leq \tau]}\} \\ &\rightarrow 0 + 0 \end{aligned}$$

as $n \rightarrow \infty$, so (a) holds.

Now we will show that (a) is equivalent to (b). Suppose that (a) holds. Then (b) also holds by the equivalence of (a) and (b) already proved. By Lemma 9.6.1 with $m = 0$ it follows that

$$|e^{it} - 1| \leq 2^{1-\delta} |t|^\delta$$

for all $t \in \mathbb{R}$ and $\delta \in [0, 1]$. We will use the two extreme cases $\delta = 0$ and $\delta = 1$. Then it follows that for each fixed $\epsilon > 0$ we have

$$\begin{aligned} |\phi_{nk}(t) - 1| &\leq E|e^{itX_{nk}} - 1| = E|e^{itX_{nk}} - 1|1_{[|X_{nk}| \leq \epsilon]} + E|e^{itX_{nk}} - 1|1_{[|X_{nk}| > \epsilon]} \\ &\leq |t|E\{|X_{nk}|1_{[|X_{nk}| \leq \epsilon]}\} + 2P(|X_{nk}| > \epsilon) \end{aligned}$$

where we used the previous display with $\delta = 1$ to handle the first term and the previous display with $\delta = 0$ to handle the second term. Thus

$$\begin{aligned} \max_{k \leq n} |\phi_{nk}(t) - 1| &\leq |t| \max_{k \leq n} E\{|X_{nk}|1_{[|X_{nk}| \leq \epsilon]}\} + 2 \max_{k \leq n} P(|X_{nk}| > \epsilon) \\ &\rightarrow 0 + 0 = 0 \end{aligned}$$

uniformly for $|t| \leq T$ by (b) and (a) respectively.

Now we show that (b) implies (a). To see this we use Inequality 9.5.1: for each $\epsilon > 0$ we have

$$\max_{k \leq n} P(|X_{nk}| > \epsilon) \leq 7\epsilon \max_{k \leq n} \int_0^{1/\epsilon} (1 - \operatorname{Re}\phi_{nk}(t)) dt \rightarrow 0$$

since (b) holds. This completes the proof of equivalence of (a), (b), and (c).

2. PfS Course Notes, Exercise 10.2.8, page 237.

(i) Show that Lindeberg's condition that $LF_n(\epsilon) \rightarrow 0$ for all $\epsilon > 0$ implies Feller's condition that $\max_{1 \leq k \leq n} \sigma_{n,k}^2 / \sigma_n^2 \rightarrow 0$.

(ii) Let X_{n1}, \dots, X_{nn} be row independent Poisson(λ/n) random variables with $\lambda > 0$. Discuss which of the Lindeberg-Feller, Liapunov, and Feller conditions holds in this context. [The Liapunov $(2 + \delta)$ condition is as follows: for some $0 < \delta \leq 1$ we have

$$\sum_{k=1}^n E|X_{nk} - \mu_{nk}|^{2+\delta} / \sigma_n^{2+\delta} \rightarrow 0.]$$

(iii) Repeat part (ii) when X_{n1}, \dots, X_{nn} are row independent and all have the probability density $cx^{-3}(\log x)^{-2}$ on $x \geq 3$ (for some constant $c > 0$).

(iv) Repeat part (ii) when $P(X_{nk} = a_k) = P(X_{nk} = -a_k) = 1/2$ for row-independent X_{nk} 's. Discuss this for general a_k 's and present two or three interesting examples for which the various conditions differ (i.e. hold or fail to hold).

Solution: (i) Suppose that $LF_n(\epsilon) \rightarrow 0$ for all $\epsilon > 0$. Fix $\epsilon > 0$. Then

$$\begin{aligned}\frac{\sigma_{nk}^2}{\sigma_n^2} &= \frac{1}{\sigma_n^2} \{EX_{nk}^2 1_{[|X_{nk}| \leq \epsilon \sigma_n]} + EX_{nk}^2 1_{[|X_{nk}| > \epsilon \sigma_n]}\} \\ &\leq \epsilon^2 + \frac{1}{\sigma_n^2} E\{X_{nk}^2 1_{[|X_{nk}| > \epsilon \sigma_n]}\}\end{aligned}$$

and hence

$$\begin{aligned}\max_{k \leq n} \frac{\sigma_{nk}^2}{\sigma_n^2} &\leq \epsilon^2 + \max_{k \leq n} \frac{1}{\sigma_n^2} E\{X_{nk}^2 1_{[|X_{nk}| > \epsilon \sigma_n]}\} \\ &\leq \epsilon^2 + \frac{1}{\sigma_n^2} \sum_{k=1}^n E\{X_{nk}^2 1_{[|X_{nk}| > \epsilon \sigma_n]}\} \\ &= \epsilon^2 + LF_n(\epsilon) \rightarrow \epsilon^2\end{aligned}$$

as $n \rightarrow \infty$. Since $\epsilon > 0$ is arbitrary we conclude that

$$\max_{k \leq n} \frac{\sigma_{nk}^2}{\sigma_n^2} \rightarrow 0.$$

(ii) When the X_{nk} 's are i.i.d. $\text{Poisson}(\lambda/n)$, then $\mu_{nk} = \lambda/n = \sigma_{nk}^2$, so $\sum_{k=1}^n \mu_{nk} = n(\lambda/n) = \lambda$ and $\sum_{k=1}^n \sigma_{nk}^2 = \lambda$. Since $\sum_{k=1}^n X_{nk} \stackrel{d}{=} N_\lambda \sim \text{Poisson}(\lambda)$, we have

$$Z_n \equiv \frac{1}{\sigma_n} \sum_{k=1}^n (X_{nk} - \mu_{nk}) \stackrel{d}{=} \frac{N_\lambda - \lambda}{\sqrt{\lambda}}$$

does not converge in distribution to $N(0, 1)$. This implies that the Lindeberg condition fails (since if it holds then it would follow that $Z_n \rightarrow_d Z \sim N(0, 1)$). It further follows that all the Liapunov- $2 + \delta$ conditions fail, since they all imply that the Lindeberg condition holds. Here the Feller condition

$$\max_{k \leq n} \frac{\sigma_{nk}^2}{\sigma_n^2} = \frac{\lambda/n}{\lambda} = \frac{1}{n} \rightarrow 0$$

holds.

(iii) If $f(x) = cx^{-3}(\log x)^{-2} 1_{[e, \infty)}(x)$, then

$$\begin{aligned}EX^2 &= \int_e^\infty cx^2 x^{-3} (\log x)^{-2} dx = c \int_e^\infty x^{-1} (\log x)^{-2} dx \\ &= c \int_1^\infty v^{-2} dv = c < \infty\end{aligned}$$

by the change of variable $v = \log x$. Thus $\sigma^2 = \text{Var}(X_{nk}) < \infty$ and $EX_{nk} = \mu$ is well-defined. Hence $\sigma_n^2 = n\sigma^2$ and $\max_{k \leq n} \sigma_{nk}^2 / \sigma_n^2 = 1/n \rightarrow 0$, so the Feller condition holds. Since $\sigma^2 < \infty$, we know from the ordinary CLT that

$$\frac{\sum_{k=1}^n (X_{nk} - \mu_{nk})}{\sigma_n} = \sqrt{n}(\bar{X}_n - \mu) / \sigma \rightarrow_d Z \sim N(0, 1).$$

Thus by the Lindeberg-Feller CLT the Lindeberg condition holds. This can also be checked directly since

$$LF_n(\epsilon) = \frac{1}{n\sigma^2} n E(X - \mu)^2 1_{\{|X - \mu| > \epsilon \sigma \sqrt{n}\}} = \frac{1}{\sigma^2} E(X - \mu)^2 1_{\{|X - \mu| > \epsilon \sigma \sqrt{n}\}} \rightarrow 0$$

for every $\epsilon > 0$ by the Dominated Convergence Theorem (with dominating function $(X - \mu)^2$). On the other hand all the Liapunov $2 + \delta$ conditions fail: Note that by the C_r inequality we have, with $r = 2 + \delta$,

$$|x|^r \leq C_r |x - \mu|^r + |\mu|^r,$$

so that $|x - \mu|^{2+\delta} \geq |x|^{2+\delta} / C_{2+\delta} - |\mu|^{2+\delta}$. Thus

$$\begin{aligned} E|X - \mu|^{2+\delta} &\geq E|X|^{2+\delta} / C_{2+\delta} - \mu^{2+\delta} \\ &= \int_e^\infty x^{2+\delta} f(x) / C_{2+\delta} - \mu^{2+\delta} \\ &= c \int_e^\infty x^{-1+\delta} (\log x)^{-2} dx - \mu^{2+\delta} = \infty. \end{aligned}$$

(iv) In this case $X_k \stackrel{d}{=} a_k \epsilon_k$ where the $\{\epsilon_k\}$ are i.i.d. Rademacher random variables. Since $\sigma_{nk}^2 = a_k^2$, the Feller (uan) condition becomes $\max_{1 \leq k \leq n} \sigma_{n,k}^2 / \sigma_n^2 = \max_{1 \leq k \leq n} a_k^2 / \sum_{k=1}^n a_k^2 \rightarrow 0$. From (i) above we know that this is implied by the Lindeberg-Feller condition. On the other hand, it is always true that the Liapunov $2 + \delta$ condition implies the Lindeberg-Feller condition:

$$\begin{aligned} LF_n(\epsilon) &= \frac{1}{\sigma_n^2} \sum_{k=1}^n E \left\{ |X_{n,k} - \mu_{n,k}|^2 1_{\{|X_{n,k} - \mu_{n,k}| > \epsilon \sigma_n\}} \right\} \\ &\leq \frac{1}{\sigma_n^2} \sum_{k=1}^n E \left\{ \frac{|X_{n,k} - \mu_{n,k}|^{2+\delta}}{\epsilon^\delta \sigma_n^\delta} 1_{\{|X_{n,k} - \mu_{n,k}| > \epsilon \sigma_n\}} \right\} \\ &\leq \frac{1}{\epsilon^\delta \sigma_n^{2+\delta}} \sum_{k=1}^n E |X_{n,k} - \mu_{n,k}|^{2+\delta} \\ &\rightarrow 0 \end{aligned}$$

where the last line follows if the Liapunov $2 + \delta$ condition holds. But now we will show that in the present case the Liapunov $2 + \delta$ condition holds if the Feller

(uan) condition holds, and thus all three conditions are equivalent in this case. In the present case we have

$$\begin{aligned} \frac{1}{\sigma_n^{2+\delta}} \sum_{k=1}^n E|X_{n,k} - \mu_{n,k}|^{2+\delta} &= \frac{1}{(\sum_{k=1}^n a_k^2)^{(2+\delta)/2}} \sum_{k=1}^n |a_k|^{2+\delta} \\ &\leq \frac{\max_{1 \leq k \leq n} |a_k|^\delta}{(\sum_{k=1}^n a_k^2)^{\delta/2}} = \left(\frac{\max_{1 \leq k \leq n} a_k^2}{\sum_{k=1}^n a_k^2} \right)^{\delta/2} \\ &\rightarrow 0 \end{aligned}$$

if Feller's uan condition holds.

3. Suppose that $\{X_k : 1 \leq k < \infty\}$ are independent random variables with $P(X_k = \pm k) = 1/(2k^2)$ and (for $k \geq 2$) $P(X_k = \pm 1) = (1 - k^{-2})/2$. Let $S_n = \sum_{k=1}^n X_k$.
- Show that $\text{Var}(S_n)/n \rightarrow 2$.
 - Compute $\max_{1 \leq k \leq n} \text{Var}(X_k)/\text{Var}(S_n)$ and show that it converges to 0.
 - Does the Lindeberg-Feller condition hold?
 - Does $S_n/\sqrt{\text{Var}(S_n)} \rightarrow N(0, 1)$?

Solution: (a) Now $E(X_k) = 0$ for all $k \geq 1$ and hence $\text{Var}(X_1) = 1$ while, for $k \geq 2$,

$$\text{Var}(X_k) = E(X_k^2) = 1^2(1 - k^{-2}) + k^2 \cdot k^{-2} = 2 - k^{-2}.$$

This yields $\text{Var}(S_n) = 1 + \sum_{k=2}^n \text{Var}(X_k) = 1 + 2(n-1) - \sum_{k=2}^n k^{-2}$, and hence (noting that $\sum_{k=2}^n k^{-2} \rightarrow \pi^2/6 - 1$),

$$\text{Var}(S_n)/n = n^{-1} + 2 \frac{n-1}{n} - n^{-1} \sum_{k=1}^n k^{-2} \rightarrow 2 \text{ as } n \rightarrow \infty.$$

(b) It follows from (a) that $\max_{1 \leq k \leq n} \text{Var}(X_k) = 2 - n^{-2}$. Therefore

$$\frac{\max_{1 \leq k \leq n} \text{Var}(X_k)}{\text{Var}(S_n)} = \frac{2 - n^{-2}}{1 + 2(n-1) + n^{-1} \sum_{k=1}^n k^{-2}} \rightarrow 0.$$

(c) The Lindeberg(-Feller) condition necessarily fails: with $\sigma_n^2 \equiv \text{Var}(S_n)$ we have

$$\begin{aligned} S_n/\sigma_n &= \frac{\sum_{k=1}^n X_k 1_{\{|X_k| \leq 1\}} + \sum_{k=1}^n X_k 1_{\{|X_k| > 1\}}}{\sigma_n} \\ &= \frac{\sqrt{n} \sum_{k=1}^n X_k 1_{\{|X_k| \leq 1\}}}{\sigma_n \sqrt{n}} + \frac{\sum_{k=1}^n X_k 1_{\{|X_k| > 1\}}}{\sigma_n} \\ &\equiv a_n Z_n + B_n \\ &\rightarrow_d \frac{1}{\sqrt{2}} Z + 0 \sim N(0, 1/2) \end{aligned}$$

by Slutsky's theorem since:

$$a_n = \sqrt{n}/\sigma_n \rightarrow 1/\sqrt{2} \text{ by (a);}$$

$Z_n = n^{-1/2} \sum_{k=1}^n U_k$ where $U_k \equiv X_k 1_{[|X_k| \leq 1]}$ has $U_1 \stackrel{d}{=} \text{Rademacher}$, and, for $k \geq 2$ $P(U_k = \pm 1) = (1 - k^{-2})/2$ and $P(U_k = 0) = k^{-2}$. Thus the U_k 's are Khinchine equivalent to i.i.d. Rademacher random variables.

$B_n = \sigma_n^{-1} \sum_{k=1}^n V_k \rightarrow_{a.s.} 0$ since $V_k \equiv X_k 1_{[|X_k| > 1]}$ satisfy $P(|V_k| > 0 \text{ i.o.}) = 0$ and $\sigma_n \sim \sqrt{2n} \rightarrow \infty$.

4. Suppose that $\{X_k : k \geq 1\}$ are independent random variables with

$$P(X_k = \pm k^\alpha) = \frac{1}{6k^{2(\alpha-1)}} \quad \text{and} \quad P(X_k = 0) = 1 - \frac{1}{3k^{2(1-\alpha)}}.$$

Show that the Lindeberg condition holds if and only if $\alpha < 3/2$.

Solution: By symmetry we see that $E(X_k) = 0$; then

$$\sigma_{n,k}^2 = \text{Var}(X_k) = E(X_k^2) = k^{2\alpha} \frac{1}{3k^{2(\alpha-1)}} = \frac{k^2}{3},$$

and it follows that

$$\sigma_n^2 = \sum_{k=1}^n \sigma_{n,k}^2 = 3^{-1} \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{3 \cdot 6} \sim \frac{n^3}{9}.$$

Thus we find that for $\epsilon > 0$

$$\begin{aligned} LF_n(\epsilon) &= \frac{1}{\sigma_n^2} \sum_{k=1}^n E\{|X_k|^2 1_{[|X_k| > \epsilon \sigma_n]}\} \\ &= \frac{3 \cdot 6}{n(n+1)(2n+1)} \sum_{k=1}^n \frac{2k^{2\alpha}}{6k^{2(\alpha-1)}} 1_{\{k^\alpha > \epsilon \sqrt{n(n+1)(2n+1)/18}\}} \\ &= \frac{6}{n(n+1)(2n+1)} \sum_{k=k_n(\epsilon)+1}^n k^2 \\ &= \frac{6}{n(n+1)(2n+1)} \left\{ \frac{n(n+1)(2n+1)}{6} - \frac{k_n(k_n+1)(2k_n+1)}{6} \right\} 1_{\{k_n < n\}} \\ &= \left\{ 1 - \frac{k_n(k_n+1)(2k_n+1)}{n(n+1)(2n+1)} \right\} 1_{\{k_n < n\}} \end{aligned} \quad (1)$$

where $k_n \equiv \lfloor (\epsilon/\sqrt{18})^{1/\alpha} (n(n+1)(2n+1))^{1/(2\alpha)} \rfloor$. But $k_n \sim cn^{3/(2\alpha)} \rightarrow \infty$ if $\alpha < 3/2$, so the indicator function on the right side in the last display becomes 0 when n is so large that $k_n \geq n$. On the other hand, if $\alpha = 3/2$, then $k_n \sim cn$ and the right side of 1 converges to $(1 - c^3)$; and if $\alpha > 3/2$, then $k_n/n \rightarrow 0$ and the right side of 1 converges to 1. In either of these latter two cases the Lindeberg condition fails.

5. $\{X_k : k \geq 1\}$ satisfies a Lindeberg condition of order r if

$$\frac{1}{s_n^r} \sum_{k=1}^n E\{|X_k|^r 1_{\{|X_k| > \epsilon s_n\}}\} \rightarrow 0$$

for every $\epsilon > 0$ where $s_n^2 \equiv \sum_{k=1}^n \sigma_k^2$. Suppose that $\{X_k : k \geq 1\}$ are independent random variables with $E(X_k) = 0$, $E(X_k^2) = \sigma_k^2 < \infty$. Show that if $\{X_k\}$ satisfies a Lindeberg condition of order r for some integer $r \geq 2$, then $E(S_n/s_n)^k \rightarrow EZ^k$ for each $k = 1, 2, \dots, r$ where $Z \sim N(0, 1)$.

Solution: First note that if $\{X_k\}$ satisfies the Lindeberg condition of order 2, then $1 = E(S_n/s_n)^2 \rightarrow 1 = E(Z^2)$, so the claim holds for $r = 2$. Now suppose that $r > 2$. We proceed by induction: Suppose that claim holds for $k \in \{2, \dots, r-1\}$. Then, since the Lindeberg condition of order k holds for $k \in \{2, \dots, r-1\}$, we have $E|S_n/s_n|^k = O(1)$ for $k \in \{2, \dots, r-1\}$. Now let $f_k(z) = z^k$, and consider the swapping argument used in Lindeberg's proof of the Lindeberg-Lévy CLT: Let Y_1, \dots, Y_n be independent Gaussian random variables, $Y_j \sim N(0, \sigma_j^2)$, all independent of the X_j 's. Set $W_{n,j} \equiv \sum_{i=1}^{j-1} X_i + \sum_{i=j+1}^n Y_i$. Then

$$W_{n,j} + X_j = W_{n,j+1} + Y_{j+1}, \quad 1 \leq j \leq n-1.$$

Note that with $Z_n \equiv S_n/s_n$ and $Z \sim N(0, 1)$, by the telescoping argument,

$$\begin{aligned} Ef_r(Z_n) - Ef_r(Z) &= Ef_r(W_{n,n}/\sigma_n) - Ef_r(W_{n,0}/\sigma_n) \\ &= E \sum_{j=1}^n \left\{ f_r \left(\frac{W_{n,j} + X_j}{s_n} \right) - f_r \left(\frac{W_{n,j} + Y_j}{s_n} \right) \right\} \\ &= \sum_{j=1}^n E \sum_{k=1}^{r-1} \frac{X_j^k - Y_j^k}{s_n^k k!} f_r^{(k)} \left(\frac{W_{n,j}}{s_n} \right) \\ &= \sum_{k=3}^{r-1} \sum_{j=1}^n E \frac{X_j^k - Y_j^k}{k! s_n^k} \cdot Ef_r^{(k)} \left(\frac{W_{n,j}}{s_n} \right) \end{aligned}$$

since, for $k = 1, 2$, by independence,

$$E \frac{X_j^k - Y_j^k}{s_n^k k!} \cdot f_r^{(k)} \left(\frac{W_{n,j}}{s_n} \right) = E \frac{X_j^k - Y_j^k}{s_n^k k!} \cdot Ef_r^{(k)} \left(\frac{W_{n,j}}{s_n} \right) = 0,$$

and, for $k \in \{3, \dots, r-1\}$, again by independence, and using the fact that

$\{|S_k/s_k|^\gamma : 1 \leq k \leq n\}$ is a sub-martingale for any $\gamma \geq 1$,

$$\begin{aligned}
E \frac{X_j^k - Y_j^k}{s_n^k k!} \cdot E f_r^{(k)} \left(\frac{W_{n,j}}{s_n} \right) &= E \frac{X_j^k - Y_j^k}{s_n^k k!} \cdot \frac{r!}{(r-k)!} E \left\{ \left(\frac{W_{n,j}}{s_n} \right)^{r-k} \right\} \\
&= \binom{r}{k} E \frac{X_j^k - Y_j^k}{s_n^k} \{ E |S_n/s_n|^{r-k} + E |Z|^{r-k} \} \\
&= O(1) E \left\{ \frac{X_j^k - Y_j^k}{s_n^k} \right\}.
\end{aligned}$$

Substituting the resulting identity in (2) and noting that

$$\begin{aligned}
E \left(\sum_{j=1}^n X_j^k / s_n^k \right) &= E (S_n / s_n)^k = E (Z_n^k), \quad \text{and} \\
E \left(\sum_{j=1}^n Y_j^k / s_n^k \right) &= E \left(\sum_{j=1}^n Y_j / s_n \right)^k = E Z^k,
\end{aligned}$$

yields

$$\begin{aligned}
|E f(Z_n) - E f(Z)| &\leq O(1) \sum_{k=3}^{r-1} |E f_k(Z_n) - E f_k(Z)| \\
&\rightarrow 0
\end{aligned}$$

by the induction hypothesis.