Statistics 523, Problem Set 1 Solutions

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1. State the tentative topic or title of your Project / Paper and several key references. A tentative outline will be due on Wednesday, May 3; the Project / Paper itself will be due on June 5.

2. PfS Course Notes, Exercise 9.3.5; PfS (2000), Exercise 13.1.4, page 371. Show that the real part of a characteristic function (or $Re\phi(\cdot)$) is itself a characteristic function.

   Solution: Note that since $\overline{\phi}_X = \overline{\phi}_{-X}$ we have

$$\begin{aligned}
\mathrm{Re}\phi_X(t) &= \frac{1}{2}(\phi_X(t) + \overline{\phi}_X(t)) = \frac{1}{2}(\phi_X(t) + \phi_{-X}(t)) \\
&= \frac{1}{2}\left(Ee^{itX} + Ee^{-itX}\right) = \frac{1}{2}\left(\int e^{itx} dF_X(x) + \int e^{itx} dF_{-X}(x)\right) \\
&= \int e^{itx} d(F_X(x) + F_{-X}(x))/2 \equiv \int e^{itx} dG(x)
\end{aligned}$$

   where $G(x) \equiv (1/2)(F_X(x) + F_{-X}(x))$ is the distribution function of ϵX where ϵ is a Rademacher random variable independent of X.

3. PfS Course Notes, Exercise 9.3.6; PfS (2000), Exercise 13.1.5, page 371. Let ϕ be a chf. Show that $c^{-1}\int_0^c \phi(tu)du$ is a chf.

   Solution: Let U be a Uniform$(0, c)$ random variable independent of X. Then let $Y \equiv UX$. The characteristic function of Y is

$$\begin{aligned}
\phi_Y(t) &= Ee^{itY} = Ee^{itUX} = E\{E\{e^{itUX}|U\}\} \\
&= E\{\phi_X(tU)\} = \frac{1}{c}\int_0^c \phi_X(tu)du.
\end{aligned}$$

   Thus if ϕ is the characteristic function of X, then the given expression is the characteristic function of UX where $U \sim$ Uniform$(0, c)$ is independent of X.

   In fact a random variable Y has a unimodal distribution if and only if it has a characteristic function of the form given in the display with

$c = 1$ (and hence also if and only if $Y = UX$ for $U \sim \text{Uniform}(0,1)$ and $X \sim F$); see e.g. Dharmadhikari,and Joag-Dev (1988), *Unimodality, Convexity, and Applications*, page 7.

4. PfS Course Notes, Exercise 9.3.3 (PfS (2000), Exercise 13.1.3(c), page 345).
   Derive the Logistic$(0,1)$ characteristic function. Hint: use lemma 3.2.

   Solution: The logistic density is given by $f(x) = e^{-x}/(1+e^{-x})^2$, so we want to calculate

$$\phi(t) = Ee^{itX} = \int_{-\infty}^{\infty} \frac{e^{itx}e^{-x}}{(1+e^{-x})^2} dx.$$

   We want to show that this equals $\pi t/\sinh(\pi t)$ for $t \in \mathbb{R}$. Consider the function ϕ as a function of a complex variable z: thus

$$\phi(z) = \int_{-\infty}^{\infty} \frac{e^{izx}e^{-x}}{(1+e^{-x})^2} dx.$$

   Also consider the function $\psi(z) = \pi z/\sinh(\pi z)$. Now both ϕ and ψ are analytic functions of $z = x + iy$ for $|Im(z)| < 1$. Note that

$$\psi(iy) = \frac{\pi(iy)}{\sinh(\pi(iy))} = \frac{\pi iy}{i\sin(\pi y)} = \frac{\pi y}{\sin(\pi y)}$$

   so that when $y = \pm 1$,

$$\psi(\pm i) = \frac{\pm\pi}{\sin(\pm\pi)} = \pm\infty.$$

   But they are both analytic on the strip $D = \{z: \ |Im(z)| < 1\}$. For ϕ on the imaginary axis we use the change of variables $(1+e^{-x}) = v$, so that $e^{-x} = (1-v)/v$, to find that

$$\begin{aligned}
\phi(iy) &= \int_{-\infty}^{\infty} \frac{e^{-yx}e^{-x}}{(1+e^{-x})^2} dx = \int_{-\infty}^{\infty} e^{-yx} d\{(1+e^{-x})^{-1}\} \\
&= \int_0^1 \left(\frac{1-v}{v}\right)^y dv = \int_0^1 v^{-y}(1-v)^y dv = \frac{\Gamma(1-y)\Gamma(y+1)}{\Gamma(2)} \\
&= \Gamma(1-y)\Gamma(y)y = \frac{\pi y}{\sin(\pi y)} = \psi(iy)
\end{aligned}$$

for $y \in (-1,1)$. Here we used the "duplication formula for the Gamma function",

$$\Gamma(1-y)\Gamma(y) = \frac{\pi}{\sin(\pi y)},$$

in the last step. Since $S = \{z: \ z = iy, \ |y| < 1\}$ has an accumulation point in D, it follows from Lemma 9.3.2 that $\phi(z) = \psi(z)$ for $z \in D$. But this implies that $\psi(t) = \phi(t)$ for $t \in \mathbb{R}$; i.e. the chf ϕ of the logistic distribution is $\pi t/\sinh(\pi t)$ for $t \in \mathbb{R}$.

5. Give an alternative derivation of the characteristic function of a Cauchy random variable X along the following lines:
(a) Let Y_1, Y_2 be independent exponential(1) random variables. Show that $V \equiv Y_1 - Y_2$ has characteristic function $\phi_V(t) = 1/(1+t^2)$.
(b) Since $|\phi_V(t)|$ is integrable, the density of V is

$$f_V(v) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{1+t^2} e^{-itv} dt \quad \text{for } v \in \mathbb{R}.$$

(c) Use the convolution formula to show that $f_V(v) = (1/2)\exp(-|v|)$.
(d) Combine (a) - (c) to conclude that $\phi_X(t) = (1/2)\exp(-|t|)$.

Solution: (a) If Y_1 and Y_2 are independent exponential random variables, then $\phi_{Y_j}(t) = (1-it)^{-1}$ and hence $V = Y_1 - Y_2$ has characteristic function

$$\begin{aligned} \phi_V(t) &= Ee^{itV} = Ee^{it(Y_1-Y_2)} = Ee^{itY_1} \cdot Ee^{-itY_2} \\ &= (1-it)^{-1}(1+it)^{-1} = (1+t^2)^{-1}. \end{aligned}$$

(b) Since $|\phi_V| = \phi_V$ is integrable, the density of V is given by

$$f_V(v) = \frac{1}{2\pi}\int_{\mathbb{R}} \phi_V(u) e^{-iuv} du = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{1+u^2} e^{-ivu} du.$$

(c) But by the convolution formula, with $f_1 = f_2 =$ the exponential(1) density,

$$\begin{aligned} f_V(v) &= \int_{\mathbb{R}} f_1(v+u) f_2(u) du = \int_0^\infty e^{-u} e^{-(v+u)} 1_{(0,\infty)}(v+u) du \\ &= \int_{\max\{0,-v\}}^\infty e^{-u} e^{-(v+u)} du = e^{-v} \int_{\max\{0,-v\}}^\infty e^{-2u} du \\ &= \begin{cases} e^{-v} \cdot (1/2), & \text{if } v \geq 0, \\ e^{v} \cdot (1/2), & \text{if } v < 0, \end{cases} \\ &= 2^{-1} e^{-|v|}. \end{aligned}$$

(d) Combining (b) and (c) yields

$$\frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{1+u^2}e^{-ivu}du = 2^{-1}e^{-|v|}.$$

Multiplication by 2 and noting that the Cauchy density is $\pi^{-1}(1+x^2)^{-1}$, yields the claim: $\phi_X(t) = e^{-|t|}$.