Statistics 523, Problem Set 6

Wellner; 5/3/2017

Reading:	Shorack, PfS Course Notes, Chapter 12, pages 312-332; Durrett, PTE, Chapter 8, pages 353 - 400.
Due:	Wednesday, May 10, 2017.
Reminder:	Midterm exam, Friday, May 12, 2017

1. Find a random variable Y with distribution function F having $EY^2 = \infty$ but with $F \in \mathcal{D}(Normal)$.

2. Use a reflection principle to show that for $0 \le y \le x$

$$P(\sup_{0\le s\le t} \mathbb{S}(s) \ge x,\ \mathbb{S}(t) \le y) = P(\mathbb{S}(t) \ge 2x - y)\,,$$

and use this to show that the joint density of $M^+ \equiv \sup_{0\le s\le t} \mathbb{S}(s)$ and $\mathbb{S}(t)$ is given by

$$f(x,y) = \sqrt{\frac{2}{\pi t^3}}(2x-y)\exp\left(-\frac{(2x-y)^2}{2t}\right) \qquad \text{for } \ 0 \le y \le x\,.$$

Check to make sure that this gives the correct marginal densities.

3. PfS Course Notes, Exercise 12.9.2, page 332:
Let $V_n(r) \equiv \sum_{k=1}^n |\mathbb{S}(t_{nk}) - \mathbb{S}(t_{n,k-1})|^r$ and $\|\mathcal{P}_n\| \equiv \sup_{1\le k\le n} |t_{nk} - t_{n,k-1}|$.
(α) Let $Z \sim N(0,1)$. Let $r > 0$. Show that:
(a) $C_r \equiv E|Z|^r = 2^{r/2}\Gamma((r+1)/2)/\sqrt{\pi}$.
(b) $E|\mathbb{S}(t_{n,k-1}, t_{n,k}]|^r = C_r|t_{n,k} - t_{n,k-1}|^{r/2}$,
$Var(|\mathbb{S}(t_{n,k-1}, t_{n,k}]|^r) = (C_{2r} - C_r^2)|t_{n,k} - t_{n,k-1}|^r$.
(β) Now show that $EV_n(2) = 1$ and $Var(V_n(2)) \le (C_{2r} - C_r^2)\|\mathcal{P}_n\|$ and hence

$$\sum_{n=1}^\infty P(|V_n(2) - 1| \ge \epsilon) \le \epsilon^{-2}(C_{2r} - C_r^2)\sum_{n=1}^\infty \|\mathcal{P}_n\|.$$

(γ) Finally prove that $V_n(2) \to_{a.s.} 1$ if $\sum_{n=1}^\infty \|\mathcal{P}_n\| < \infty$.

4. Let $\tau = \tau_{ab}$ of Theorem 12.6.1; i.e. $\tau_{a,b} = \inf\{t : \mathbb{S}(t) \in (-a,b)^c\}$ for $a, b > 0$. Show that $E(\tau^2) \le 4ab(a+b)^2$. (This is slightly different from the statement of (4) in Theorem 12.6.1 on page 319, but seems to be consistent with (5) on the same page with $r = 2$. My current computation yields, in fact, $E(\tau^2) \le 2ab(a+b)^2$. *Hint:* use the martingale $\mathbb{S}^4(t) - 6t\mathbb{S}^2(t) + 3t^2$ (which follows from considering the 4th derivative with respect to θ of the exponential martingale $V_\theta(t) = \exp(\theta\mathbb{S}(t) - \theta^2 t^2/2)$ at $\theta = 0$; see Exercise 12.7.3, page 325.

5. **Optional Bonus problem 1:** PfS Course Notes, Exercise 12.9.3, page 332: Prove theorem 9.1(b) when all $t_{nk} = k/2^n$. That is, with $V_n(r) \equiv \sum_{k=1}^n |\mathbb{S}(t_{nk}) - \mathbb{S}(t_{n,k-1})|^r$, show that $V_n(1) \to_{a.s.} \infty$ if $\|\mathcal{P}_n\| \equiv \sup_{1\le k\le n} |t_{nk} - t_{n,k-1}| \to 0$.

6. **Optional bonus problem 2:** Exercise 12.7.1, PfS Course notes, page 323. That is, prove that

$$P(\|\mathbb{S}\|_0^1 > a) = 1 - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{(2k+1)^2\pi^2}{8a^2}\right).$$

See Chung (1974), page 223. This yields the "small ball probability"

$$\begin{aligned} P(\|\mathbb{S}\|_0^1 \le a) &= \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{(2k+1)^2\pi^2}{8a^2}\right) \\ &\le \frac{4}{\pi}\exp(-\pi^2/(8a^2)) \end{aligned}$$

which converges to zero exponentially fast as $a \to 0$, and is a general phenomena associated with Gaussian processes: see Ledoux and Talagrand (1991), pages 60-61 and 289-290.