

Statistics 523, Problem Set 6, revised

Wellner; 5/3/2017

Reading: Shorack, PfS Course Notes, Chapter 12, pages 312-332;
Durrett, PTE, Chapter 8, pages 353 - 400.

Due: Wednesday, May 10, 2017.

Reminder: Midterm exam, Friday, May 12, 2017

1. Find a random variable Y with distribution function F having $EY^2 = \infty$ but with $F \in \mathcal{D}(\text{Normal})$.
2. Use a reflection principle to show that for $0 \leq y \leq x$

$$P\left(\sup_{0 \leq s \leq t} \mathbb{S}(s) \geq x, \mathbb{S}(t) \leq y\right) = P(\mathbb{S}(t) \geq 2x - y),$$

and use this to show that the joint density of $M^+ \equiv \sup_{0 \leq s \leq t} \mathbb{S}(s)$ and $\mathbb{S}(t)$ is given by

$$f(x, y) = \sqrt{\frac{2}{\pi t^3}}(2x - y) \exp\left(-\frac{(2x - y)^2}{2t}\right) \quad \text{for } 0 \leq y \leq x.$$

Check to make sure that this gives the correct marginal densities.

3. PfS Course Notes, Exercise 12.9.2, page 332:
Let $V_n(r) \equiv \sum_{k=1}^n |\mathbb{S}(t_{nk}) - \mathbb{S}(t_{n,k-1})|^r$ and $\|\mathcal{P}_n\| \equiv \sup_{1 \leq k \leq n} |t_{nk} - t_{n,k-1}|$.
(α) Let $Z \sim N(0, 1)$. Let $r > 0$. Show that:
 (a) $C_r \equiv E|Z|^r = 2^{r/2} \Gamma((r+1)/2) / \sqrt{\pi}$.
 (b) $E|\mathbb{S}(t_{n,k-1}, t_{n,k})|^r = C_r |t_{n,k} - t_{n,k-1}|^{r/2}$,
 $Var(|\mathbb{S}(t_{n,k-1}, t_{n,k})|^r) = (C_{2r} - C_r^2) |t_{n,k} - t_{n,k-1}|^r$.
 (β) Now show that $EV_n(2) = 1$ and $Var(V_n(2)) \leq (C_{2r} - C_r^2) \|\mathcal{P}_n\|$
and hence

$$\sum_{n=1}^{\infty} P(|V_n(2) - 1| \geq \epsilon) \leq \epsilon^{-2} (C_{2r} - C_r^2) \sum_{n=1}^{\infty} \|\mathcal{P}_n\|.$$

- (γ) Finally prove that $V_n(2) \rightarrow_{a.s.} 1$ if $\sum_{n=1}^{\infty} \|\mathcal{P}_n\| < \infty$.

4. **Optional Bonus problem 1:** PfS Course Notes, Exercise 12.9.3, page 332: Prove theorem 9.1(b) when all $t_{nk} = k/2^n$. That is, with $V_n(r) \equiv \sum_{k=1}^n |\mathbb{S}(t_{nk}) - \mathbb{S}(t_{n,k-1})|^r$, show that $V_n(1) \rightarrow_{a.s.} \infty$ if $\|\mathcal{P}_n\| \equiv \sup_{1 \leq k \leq n} |t_{nk} - t_{n,k-1}| \rightarrow 0$.
5. **Optional bonus problem 2:** Exercise 12.7.1, PfS Course notes, page 323. That is, prove that

$$P(\|\mathbb{S}\|_0^1 > a) = 1 - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{(2k+1)^2 \pi^2}{8a^2}\right).$$

See Chung (1974), page 223. This yields the “small ball probability”

$$\begin{aligned} P(\|\mathbb{S}\|_0^1 \leq a) &= \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{(2k+1)^2 \pi^2}{8a^2}\right) \\ &\leq \frac{4}{\pi} \exp(-\pi^2/(8a^2)) \end{aligned}$$

which converges to zero exponentially fast as $a \rightarrow 0$, and is a general phenomena associated with Gaussian processes: see Ledoux and Talagrand (1991), pages 60-61 and 289-290.

6. **Optional bonus problem 3:** Let $\tau = \tau_{ab}$ of Theorem 12.6.1; i.e. $\tau_{a,b} = \inf\{t : \mathbb{S}(t) \in (-a, b)^c\}$ for $a, b > 0$. It was shown in class that $E(\tau^2) \leq 4ES^4(\tau) \leq 4ab(a+b)^2$. Show that for any $r \geq 1$ the following inequalities hold:

$$E(\tau^r) \leq r\Gamma(r)2^{2r} ES^{2r}(\tau) \leq r\Gamma(r)2^{2r} ab(a+b)^{2r-2}.$$