Statistics 523, Problem Set 5, revised

Wellner; 4/26/2017

Reading:	Shorack, PfS Course Notes, Chapter 12, pages 312-332; Durrett, PTE, Chapter 8, pages 353 - 396.
Reminder:	Make-up lecture 2: Wednesday, April 26, 12:30 - 1:20, CMU 230.
Reminder:	Tentative paper/project outlines due **next** Wednesday, May 3.
Reminder:	Mid-term exam: Friday, May 12.
Due:	Wednesday, May 3, 2017.

1. Suppose that $a_n \nearrow$ with $a_1 = 1$ and suppose that $a_{mk} = a_m a_k$ for all $k, m \geq 1$. Show that $a_n = n^{1/\alpha}$ for some $\alpha \geq 0$.

2. Assume that Y with distribution function G is stable with characteristic exponent α. Show that $E|Y|^r < \infty$ for all $0 < r < \alpha$. [Hint: use Theorem 11.4.1(c) to show that $x^\alpha P(|Y| > x)$ is bounded for large x. Then bound the appropriate integral.]

3. (a) Let Y be a stable random variable with $\theta = 1$ and $0 < \alpha < 1$. Show that $P(Y \geq 0) = 1$.
(b) Let Y be as in (a). By the conclusion of (a) the Laplace transform of Y, $\psi(\lambda) = E\exp(-\lambda Y)$ is well-defined. Show that $Y_1 + \cdots + Y_k \stackrel{d}{=} a_k Y + b_k$ holds with $b_k = 0$ (and $Y_1, \ldots, Y_k$ i.i.d. as Y).
(c) Show that $\psi(\lambda)^n = \psi(n^{1/\alpha}\lambda)$ and hence that $\psi(\lambda) = \exp(-c\lambda^\alpha)$ for some $c > 0$.

4. Let $X_1, X_2, \ldots$ be i.i.d. with a density function f that is symmetric about 0 and continuous and positive at 0. Show that

$$\frac{1}{n}\left(\frac{1}{X_1} + \cdots + \frac{1}{X_n}\right) \rightarrow_d Y$$

where Y has a Cauchy distribution. Hint: See Durrett, PTE, Example 3.7.1, page 165 (or page 141 of the online .pdf version).

5. (i) Suppose that X is symmetric stable with index α and $Y \geq 0$ is an independent stable with index $\beta < 1$, then $XY^{1/\alpha}$ is symmetric stable with index $\alpha\beta$. Hint: Use problem 3.
(ii) Suppose that Z_1 and Z_2 are independent $N(0,1)$ random variables. Check that $1/Z_1^2$ has the density of T_1 the first time Brownian motion starting from 0 hits the level 1. Use this (and part (i)) to show that Z_1/Z_2 has a Cauchy distribution.

6. **Bonus problem 1:** Suppose that $\{X(t) : t \geq 0\}$ is a process with stationary and independent increments with $X(0) = 0$ and characteristic function of $X(t)$given by

$$Ee^{iuX(t)} = \exp(-tc|u|^\alpha\{1 - i\text{sign}(u)C_\alpha\})$$

where $\alpha \in (0,1)$, $c \geq 0$ and $C_\alpha = \tan(\pi\alpha/2)$. Thus the (marginal, or one-dimensional) distributions of $X(t)$ are completely asymmetric stable laws with exponent $\alpha \in (0,1)$.
(a) Show that $X(t) \stackrel{d}{=} t^{1/\alpha}X(1)$ for all $t > 0$.
(b) Let $0 < r < \alpha$. Use (a) to compute $E|X(t)|^r$ in terms of $E|X(1)|^r$ where the latter is finite by Problem 2.

7. **Bonus problem 2:** Now let $\mathbb{S}$ be a standard Brownian motion on $[0,\infty)$, let $X(t)$ be a completely asymmetric stable process (sometimes called a *stable subordinator*) of index $\alpha \in (0,1)$ as in bonus problem 1 above which is independent of $\mathbb{S}$. Consider the new process $Y(t) \equiv \mathbb{S}(X(t))$ for $t \geq 0$.
(a) Use a calculation similar to that of Problem 5 to show that Y is a symmetric stable process of index 2α.
(b) Does Y have stationary independent increments?

8. **Bonus problem 3:** Assume that n objects $X_{n,1}, \ldots, X_{n,n}$ are placed independently and at random in $[-n,n]$. Let

$$Z_n = \sum_{k=1}^{n} \text{sign}(X_{n,k})/|X_{n,k}|^p$$

be the total force exerted on 0.
(i) Show that if $p > 1/2$, then $E\exp(itZ_n) \to \exp(-c|t|^{1/p})$ for $t \in \mathbb{R}$ for some c.
(ii) Show that if $p < 1/2$, then $Z_n/n^{1/2-p} \to_d cZ$ where $Z \sim N(0,1)$.
(iii) Show that if $p = 1/2$, then $Z_n/(\log n)^{1/2} \to cZ$.
Hint: see Durrett, PTE, Example 3.7, page 165, and Exercise 3.7.3 page 166.