Statistics 523, Problem Set 3

Wellner; 4/12/2017

Reading: Shorack, PfS Course Notes, Chapter 11, pages 281-287
Goldstein, *Amer. Math. Monthly* **116** (2009), 45 - 60.

Due: Wednesday, April 19, 2017.

1. (a) Give an example of a random variable Y with distribution function F on $\mathbb{R}^+ = [0, \infty)$ for which $EY^r = \infty$ for all $r > 0$.
(b) Does your example in (a) have $Eg(Y) < \infty$ for some measurable function g with $g(y) \to \infty$ as $y \to \infty$.

2. A very famous theorem conjectured by Lévy and proved by Cramér (1936) says that if X and Y are independent random variables with $X + Y = Z$ having a Normal distribution, then both X and Y have normal distributions. Find a statement and proof of this theorem. What are the crucial ingredients of the proof?

3. Stein's method for convergence in distribution to the Poisson distribution depends on the following characterization: $X \sim$ Poisson(λ) if and only if

$$E[Xf(X)] = \lambda E[f(X + 1)]$$

for all functions f for which the expectations exist. Show that if $X \sim$ Poisson(λ) then the identity in the display holds for any bounded function f.

4. Goldstein's probabilistic proof of the Lindeberg-Feller CLT relies on the following lemma, which is a kind of converse for Slutsky's lemma.
Let $\{U_n\}$ and $\{V_n\}$ be sequences of random variables such that U_n and V_n are independent for every n. Then $U_n \to U$ and $U_n + V_n \to_d U$ implies that $V_n \to_p 0$. Prove this lemma. (This is Lemma 5.1 in Goldstein (2009).

5. **Optional bonus problem:** Goldstein's probabilistic proof of the Lindeberg-Feller CLT relies on a second lemma lemma concerning the stability (or preservation) of the CLT under the zero-bias transformation: If $\{Y_n\}$ is a sequence of mean zero variables with finite variances $\sigma_n^2 = Var(Y_n)$ satisfying $Y_n \to_d Y$ and $\sigma_n^2 \to \sigma^2 = Var(Y)$ where Y also has mean zero, show that $Y_n^* \to_d Y^*$ where, for any mean zero finite variance random variable W, W^* has the zero bias distribution corresponding to the distribution of W. Prove this lemma.