

Statistics 523, Problem Set 2

Wellner; 4/5/17

Reading: Shorack, PFS Course Notes, Chapter 10, pages 225-253;
Shorack, PFS Course Notes, Chapter 11, pages 273-287.

Due: Wednesday, April 12, 2017.

- PfS Course Notes, Exercise 10.2.1, page 236. (Characterization of “uan”) Suppose that $\{X_{n,k} : 1 \leq k \leq n\}$ is a row-independent triangular array with $E(X_{n,k}) = 0$, $E(X_{n,k}^2) \equiv \sigma_{n,k}^2$, normalized so that $\sigma_n^2 \equiv \sum_{k=1}^n \sigma_{n,k}^2 = 1$. Show that the following are equivalent:
 - $|X_{n,k}|$'s are uan; that is, $\max_{1 \leq k \leq n} P(|X_{n,k}| \geq \epsilon) \rightarrow 0$ for all $\epsilon > 0$.
 - $\max_{1 \leq k \leq n} |\phi_{nk}(t) - 1| \rightarrow 0$ uniformly on every finite interval of t 's.
 - $\max_{1 \leq k \leq n} E(X_{n,k}^2 \wedge 1) = \max_{1 \leq k \leq n} \int (x^2 \wedge 1) dF_{nk}(x) \rightarrow 0$.
- PfS Course Notes, Exercise 10.2.8, page 237.
 - Show that Lindeberg's condition that $LF_n(\epsilon) \rightarrow 0$ for all $\epsilon > 0$ implies Feller's condition that $\max_{1 \leq k \leq n} \sigma_{n,k}^2 / \sigma_n^2 \rightarrow 0$.
 - Let X_{n1}, \dots, X_{nn} be row independent Poisson(λ/n) random variables with $\lambda > 0$. Discuss which of the Lindeberg-Feller, Liapunov, and Feller conditions holds in this context. [The Liapunov ($2 + \delta$) condition is as follows: for some $0 < \delta \leq 1$ we have
$$\sum_{k=1}^n E|X_{nk} - \mu_{nk}|^{2+\delta} / \sigma_n^{2+\delta} \rightarrow 0.$$
 - Repeat part (ii) when X_{n1}, \dots, X_{nn} are row independent and all have the probability density $cx^{-3}(\log x)^2$ on $x \geq 3$ (for some constant $c > 0$).
 - Repeat part (ii) when $P(X_{nk} = a_k) = P(X_{nk} = -a_k) = 1/2$ for row-independent X_{nk} 's. Discuss this for general a_k 's and present two or three interesting examples for which the various conditions differ (i.e. hold or fail to hold).
- Suppose that $\{X_k : 1 \leq k < \infty\}$ are independent random variables with $P(X_k = \pm k) = 1/(2k^2)$ and (for $k \geq 2$) $P(X_k = \pm 1) = (1 - (k^{-2}))/2$. Let $S_n = \sum_{k=1}^n X_k$.
 - Show that $Var(S_n)/n \rightarrow 2$.
 - Compute $\max_{1 \leq k \leq n} Var(X_k)/Var(S_n)$ and show that it converges to 0.
 - Does the Lindeberg-Feller condition hold?
 - Does $S_n/\sqrt{Var(S_n)} \rightarrow N(0, 1)$?

4. Suppose that $\{X_k : k \geq 1\}$ are independent random variables with

$$P(X_k = \pm k^\alpha) = \frac{1}{6k^{2(\alpha-1)}} \quad \text{and} \quad P(X_k = 0) = 1 - \frac{1}{3k^{2(1-\alpha)}}.$$

Show that the Lindeberg condition holds if and only if $\alpha < 3/2$.

5. $\{X_k : k \geq 1\}$ satisfies a Lindeberg condition of order r if

$$\frac{1}{s_n^r} \sum_{k=1}^n E\{|X_k|^r 1_{\{|X_k| > \epsilon s_n\}}\} \rightarrow 0$$

for every $\epsilon > 0$ where $s_n^2 \equiv \sum_{k=1}^n \sigma_k^2$. Suppose that $\{X_k : k \geq 1\}$ are independent random variables with $E(X_k) = 0$, $E(X_k^2) = \sigma_k^2 < \infty$. Show that if $\{X_k\}$ satisfies a Lindeberg condition of order r for some integer $r \geq 2$, then $E(S_n/s_n)^k \rightarrow EZ^k$ for each $k = 1, 2, \dots, r$ where $Z \sim N(0, 1)$.

6. **Optional bonus problem 1:** Suppose that $\{X_k : k \geq 1\}$ are independent random variables with $E(X_k) = 0$, $E(X_k^2) = \sigma_k^2 < \infty$, and $S_n/s_n \rightarrow_d Z \sim N(0, 1)$, and $E\{(s_n^{-1}S_n)^{2m}\} = (2m)!/(2^m m!)$ with $s_n^2 \equiv \sum_{k=1}^n \sigma_k^2$.

(a) Show that $\{X_k : k \geq 1\}$ satisfies a Lindeberg condition of order $2m$.

(b) Suppose that $\{X_k : k \geq 1\}$ satisfies a Lindeberg condition of order $r > 2$. Show that this implies that $\sum_{k=1}^E |X_k|^r = o(s_n^r)$.

7. **Optional bonus problem 2:** Suppose that $T \sim \text{Poisson}(\lambda)$. (a) Show that $(T - \lambda)/\sqrt{\lambda} \rightarrow_d Z \sim N(0, 1)$ as $\lambda \rightarrow \infty$.

(b) Suppose that $\{X_{n,k} : k \geq 1\}$ is a triangular array of independent Poisson random variables with parameters $\{\lambda_{n,k} : k \geq 1\}$. Suppose that $\lambda_n \equiv \sum_{k=1}^n \lambda_{n,k}$ where $\lambda_n \rightarrow \infty$. In view of (a) it is natural to conjecture that $T_n \equiv \sum_{k=1}^n X_{n,k}$ satisfies $(T_n - \lambda_n)/\sqrt{\lambda_n} \rightarrow_d Z \sim N(0, 1)$. Are any other conditions needed on the $\lambda_{n,k}$'s to prove this?