Statistics 523, Problem Set 2

Wellner; 4/5/17

Reading: Shorack, PfS Course Notes, Chapter 10, pages 225-253;
Shorack, PfS Course Notes, Chapter 11, pages 273-287.

Due: Wednesday, April 12, 2017.

1. PfS Course Notes, Exercise 10.2.1, page 236. (Characterization of "uan") Suppose that $\{X_{n,k} : 1 \le k \le n\}$ is a row-independent triangular array with $E(X_{n,k}) = 0$, $E(X_{n,k}^2) \equiv \sigma_{n,k}^2$, normalized so that $\sigma_n^2 \equiv \sum_{k=1}^n \sigma_{n,k}^2 = 1$. Show that the following are equivalent:
(a) $|X_{n,k}|$'s are uan; that is, $\max_{1 \le k \le n} P(|X_{n,k}| \ge \epsilon) \to 0$ for all $\epsilon > 0$.
(b) $\max_{1 \le k \le n} |\phi_{nk}(t) - 1| \to 0$ uniformly on every finite interval of t's.
(c) $\max_{1 \le k \le n} E(X_{n,k}^2 \wedge 1) = \max_{1 \le k \le n} \int (x^2 \wedge 1) dF_{nk}(x) \to 0$.

2. PfS Course Notes, Exercise 10.2.8, page 237.
(i) Show that Lindeberg's condition that $LF_n(\epsilon) \to 0$ for all $\epsilon > 0$ implies Feller's condition that $\max_{1 \le k \le n} \sigma_{n,k}^2/\sigma_n^2 \to 0$.
(ii) Let $X_{n1}, \ldots, X_{nn}$ be row independent Poisson(λ/n) random variables with $\lambda > 0$. Discuss which of the Lindeberg-Feller, Liapunov, and Feller conditions holds in this context. [The Liapunov $(2+\delta)$ condition is as follows: for some $0 < \delta \le 1$ we have
$$\sum_{k=1}^n E|X_{nk} - \mu_{nk}|^{2+\delta}/\sigma_n^{2+\delta} \to 0.]$$
(iii) Repeat part (ii) when $X_{n1}, \ldots, X_{nn}$ are row independent and all have the probability density $cx^{-3}(\log x)^{-2}$ on $x \ge 3$ (for some constant $c > 0$).
(iv) Repeat part (ii) when $P(X_{nk} = a_k) = P(X_{nk} = -a_k) = 1/2$ for row-independent X_{nk}'s. Discuss this for general a_k's and present two or three interesting examples for which the various conditions differ (i.e. hold or fail to hold).

3. Suppose that $\{X_k : 1 \le k < \infty\}$ are independent random variables with $P(X_k = \pm k) = 1/(2k^2)$ and (for $k \ge 2$)) $P(X_k = \pm 1) = (1 - (k^{-2})/2$. Let $S_n = \sum_{k=1}^n X_k$.
(a) Show that $Var(S_n)/n \to 2$.
(b) Compute $\max_{1 \le k \le n} Var(X_k)/Var(S_n)$ and show that it converges to 0.
(c) Does the Lindeberg-Feller condition hold?
(d) Does $S_n/\sqrt{Var(S_n)} \to N(0,1)$?

4. Suppose that $\{X_k : \ k \geq 1\}$ are independent random variables with

$$P(X_k = \pm k^\alpha) = \frac{1}{6k^{2(\alpha-1)}} \quad \text{and} \quad P(X_k = 0) = 1 - \frac{1}{3k^{2(1-\alpha)}}.$$

Show that the Lindeberg condition holds if and only if $\alpha < 3/2$.

5. $\{X_k : \ k \geq 1\}$ satisfies a Lindeberg condition of order r if

$$\frac{1}{s_n^r} \sum_{k=1}^{n} E\{|X_k|^r 1_{[|X_k| > \epsilon s_n]}\} \to 0$$

for every $\epsilon > 0$ where $s_n^2 \equiv \sum_{k=1}^n \sigma_k^2$. Suppose that $\{X_k : \ k \geq 1\}$ are independent random variables with $E(X_k) = 0$, $E(X_k^2) = \sigma_k^2 < \infty$. Show that if $\{X_k\}$ satisfies a Lindeberg condition of order r for some integer $r \geq 2$, then $E(S_n/s_n)^k \to EZ^k$ for each $k = 1, 2, \ldots, r$ where $Z \sim N(0,1)$.

6. **Optional bonus problem 1:** Suppose that $\{X_k : \ k \geq 1\}$ are independent random variables with $E(X_k) = 0$, $E(X_k^2) = \sigma_k^2 < \infty$, and $S_n/s_n \to_d Z \sim N(0,1)$, and $E\{(s_n^{-1} S_n)^{2m}\} = (2m)!/(2^m m!)$ with $s_n^2 \equiv \sum_{k=1}^n \sigma_k^2$.
(a) Show that $\{X_k : \ k \geq 1\}$ satisfies a Lindeberg condition of order $2m$.
(b) Suppose that $\{X_k : \ k \geq 1\}$ satisfies a Lindeberg condition of order $r > 2$. Show that this implies that $\sum_{k=1}^{E} |X_k|^r = o(s_n^r)$.

7. **Optional bonus problem 2:** Suppose that $T \sim \text{Poisson}(\lambda)$. (a) Show that $(T - \lambda)/\sqrt{\lambda} \to_d Z \sim N(0,1)$ as $\lambda \to \infty$.
(b) Suppose that $\{X_{n,k} : \ k \geq 1\}$ is a triangular array of independent Poisson random variables with parameters $\{\lambda_{n,k} : \ k \geq 1\}$. Suppose that $\lambda_n \equiv \sum_{k=1}^n \lambda_{n,k}$ where $\lambda_n \to \infty$. In view of (a) it is natural to conjecture that $T_n \equiv \sum_{k=1}^n X_{n,k}$ satisfies $(T_n - \lambda_n)/\sqrt{\lambda_n} \to_d Z \sim N(0,1)$. Are any other conditions needed on the $\lambda_{n,k}$'s to prove this?