Statistics 523, Problem Set 1

Wellner; 3/29/2017

Reading:

Shorack, PfS Course Notes	Sections 9.3-, pages 201-224
(PfS 2000	sections 13.1-7, pages 341-364)
Shorack, PfS Course Notes	Sections 10.0 - 10.4, pages 225-252
(PfS 2000,	sections 14.0-14.3, pages 365-382).

Due: Wednesday, April 5, 2017.

1. State the tentative topic or title of your Project / Paper and several key references. A tentative outline will be due on Wednesday, May 3; the Project / Paper itself will be due on June 5.

2. PfS Course Notes, Exercise 9.3.5; PfS (2000), Exercise 13.1.4, page 371. Show that the real part of a characteristic function (or $Re\phi(\cdot)$) is itself a characteristic function.

3. PfS Course Notes, Exercise 9.3.6; PfS (2000), Exercise 13.1.5, page 371. Let ϕ be a chf. Show that $c^{-1}\int_0^c \phi(tu)du$ is a chf.

4. PfS Course Notes, Exercise 9.3.3 (PfS (2000), Exercise 13.1.3(c), page 345).
   Derive the Logistic$(0,1)$ characteristic function. Hint: use lemma 3.2.

5. Give an alternative derivation of the characteristic function of a Cauchy random variable X along the following lines:
   (a) Let Y_1, Y_2 be independent exponential(1) random variables. Show that $V \equiv Y_1 - Y_2$ has characteristic function $\phi_V(t) = 1/(1+t^2)$.
   (b) Since $|\phi_V(t)|$ is integrable, the density of V is
   $$f_V(v) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{1+t^2} e^{-itv} dt \quad \text{for } v \in \mathbb{R}.$$
   (c) Use the convolution formula to show that $f_V(v) = (1/2)\exp(-|v|)$.
   (d) Combine (a) - (c) to conclude that $\phi_X(t) = (1/2)\exp(-|t|)$.

6. **Bonus problem 1:** PfS Course Notes, Exercise 9.4.2 (PfS (2000), Exercise 13.2.2, page 348). (Don't include computation of $E|X|$.)

7. **Bonus problem 2:** It is known from Chernoff (1964) and Groeneboom (1989) that Grenander's estimator of a monotone density has a limiting distribution determined by a random variable Z with density $f_Z(z) = (1/2)g(z)g(-z)$ (with respect to Lebesgue measure λ) where g has Fourier transform $\hat{g}$ given by

$$\hat{g}(t) = \int_{-\infty}^{\infty} e^{itz} g(z) dz = \frac{2^{1/3}}{Ai(i2^{-1/3}t)}.$$

Balabdaoui and W (2014), *Bernoulli* **20**, 231-244, used results of Schoenberg (1951) and Merkes and Salmassi (1997) to show that g is a Pólya frequency function of order infinity (i.e. PF_∞) and hence that the density f_Z is log-concave. The result of Merkes and Salmassi (1997) shows that

$$Ai(z) = Ai(0)e^{-\nu z} \prod_{k=1}^{\infty}(1 + z/a_k)\exp(-z/a_k)$$

where $\{-a_k\}_{k=1}^{\infty}$ are the zeros of the Airy function Ai. Use this to show that the Fourier transform $\hat{g}$ can be related to the characteristic function of a sum of independent (centered) exponential random variables. [It was conjectured by Balabdaoui and W (2014) that the density f_Z is strongly log-concave, but this has not yet been proved.]