Statistics 523, Final Exam

Wellner; 6/9/99

Instructons:

A. This is an "in-class" exam. You may *not* use your notes or other books.

B. You must do this exam completely on your own, with absolutely **no discussion** with other students.

1. (32 points). **Define** *four* of the following five terms:
(a) An *infinitely divisible* distribution (or random variable).
(b) A *stable distribution* (or random variable).
(c) A *stopping time* relative to a filtration $\{\mathcal{A}_t : 0 \le t < \infty\}$.
(d) The *strong Markov property* of a process $\{X(t) : 0 \le t < \infty\}$.
(e) A *tight* sequence of distribution function $\{F_n\}$
or probability measures $\{P_n\}$.

2. (36 points). Give careful **statements** of *three* of the following five theorems or results:
(a) The Berry-Esseen theorem.
(b) Donsker's theorem for the partial sum process $\{\mathbb{S}_n(t) : 0 \le t \le 1\}$.
(c) Donsker's theorem for the uniform empirical process
$\{\mathbb{U}_n(t) : 0 \le t \le 1\}$.
(d) Four properties of Brownian motion $\mathbb{S}$ on $[0, \infty)$.
(e) The Cramér - Lévy continuity theorem for characteristic functions.

3. (48 points). A. Prove the following:
If $\mathbb{S}$ is standard Brownian motion on $[0, \infty)$, then

$$P(\sup_{0 \le s \le t} \mathbb{S}(s) \ge x) = 2P(\mathbb{S}(t) \ge x) = 2(1 - \Phi(x/\sqrt{t}))\,.$$

B. For $a > 0$, $b \in R$, let

$$\tau \equiv \inf\{t > 0 : \mathbb{S}(t) = a + bt\}\,.$$

Use the martingale $Y(t) = \exp(\theta \mathbb{S}(t) - \theta^2 t/2)$ to show that

$$E \exp(-\lambda\tau) = \exp(-a\{b + (b^2 + 2\lambda)^{1/2}\}) .$$

[Hint: choose $\theta = b + (b^2 + 2\lambda)^{1/2}$.]

Do **either** problem 4 **or** problem 5.

4. (42 points). Suppose that $X_1, \ldots, X_n$ are iid F_0 (continuous) and you form the statistic

$$T_n = \int_{-\infty}^{\infty} \sqrt{n} |\mathbb{F}_n(x) - F_0(x)| dF_0(x) .$$

(i) What is the limiting distribution of T_n in terms of a Brownian bridge process $\mathbb{U}$?
(ii) Is the limit the same if you replace T_n by

$$T_n = \int_{-\infty}^{\infty} \sqrt{n} |\mathbb{F}_n(x) - F_0(x)| d\mathbb{F}_n(x) .$$

Why or why not?

5. (42 points). Suppose that B is standard Brownian motion on $[0, \infty)$.
A. Since $f(x) = x^2$ is convex and B_t is a martingale, B_t^2 is a submartingale. What do we substract from B_t^2 to get a martingale? Justify your answer.
B. The Ito calculus says that for twice differentiable functions f from R to R we have

$$f(B_t) - f(0) = \int_0^t f'(B_s) dB_s + \text{something}. \tag{1}$$

What is "something" in (1)? Hint: Note that "something" should reduce to your answer in part A when $f(x) = x^2$.
C. Apply (1) to the function $f(x) = \sinh(x)$ (recalling that $\sinh(x) \equiv (e^x - e^{-x})/2$).
D. Using the result from C, write down a martingale related to $\sinh(B_t)$.

Do **either** problem 6 **or** problem 7.

6. (42 points). Prove *one* of the following two inequalities:

$$P(|X| \geq 1/\epsilon) \leq \frac{7}{\epsilon} \int_0^{\epsilon} (1 - \text{Real}\phi(t)) dt\,;$$

$$P(|X| \geq 1/\epsilon) \leq \frac{1}{2\epsilon} \int_{[|t| \leq 2\epsilon]} |1 - \phi(t)| dt\,.$$

Hints: In proving the first inequality, you may use the fact that $\inf_{|y|\geq 1}(1 - \sin(y)/y) = (1 - \sin(1)) = .1585\ldots \geq 1/7$. The second inequality was proved in the solution to problem set 3, problem 2.

7. (42 points). Suppose that you are given the law of the iterated logarithm for Brownian motion $\mathbb{S}$ at ∞:

$$\limsup_{t\to\infty} \frac{\mathbb{S}(t)}{\sqrt{2t \log\log t}} = 1 \qquad a.s. \tag{2}$$

A. Prove the *time reversal* property of Brownian motion: if $\mathbb{S}$ is standard Brownian motion, then the process $\tilde{\mathbb{S}}(t) \equiv t\mathbb{S}(1/t)$ is also standard Brownian motion.
B. Use A together with (2) to prove the LIL for Brownian motion at 0:

$$\limsup_{t\to 0} \frac{\mathbb{S}(t)}{\sqrt{2t \log\log(1/t)}} = 1 \qquad a.s. \tag{3}$$