Statistics 523, Problem Set 8 Solutions

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1. Durrett, Lemma (c), pages 46-47: write out the details of this proof to a sufficient degree of detail to understand how the notation works. The expanded proof should probably be about twice as long as the one in the book.

   Solution: If Δ and Δ' are two partitions, we call $\Delta\Delta'$ the partition obtained by taking all the points in Δ and Δ'. If we apply (a) twice and take differences we see that $Y_t \equiv Q_t^{\Delta}(X) - Q_t^{\Delta'}(X)$ is a martingale (since the difference of two martingales is a martingale). By definition of $Q_t(Y)$ and (a) it follows that $Y_t^2 - Q_t^{\Delta\Delta'}(Y)$ is a martingale, and hence

   $$E(Q_r^{\Delta}(X) - Q_r^{\Delta'}(X))^2 = EY_r^2 = EQ_r^{\Delta\Delta'}(Y).$$

   In the following we will drop the argument from $Q_r\Delta$ when it is X and we will drop the r when referring to the process $t \mapsto Q_t^{\Delta}$. Since $(a+b)^2 \le 2a^2 + 2b^2$ for all $a, b \in \mathbb{R}$, it follows that

   $$\begin{aligned} Q_r^{\Delta\Delta'}(Y) &= \sum_k (Y_{s_{k+1}} - Y_{s_k})^2, \quad \text{where } \ Y_s = Q_s^{\Delta} - Q_s^{\Delta'} \\ &= \sum_k \{Q_{s_{k+1}}^{\Delta} - Q_{s_{k+1}}^{\Delta'} - (Q_{s_k}^{\Delta} - Q_{s_k}^{\Delta'})\}^2 \\ &= \sum_k \{(Q_{s_{k+1}}^{\Delta}) - Q_{s_k}^{\Delta} - (Q_{s_{k+1}}^{\Delta'} - Q_{s_k}^{\Delta'})\}^2 \\ &= \sum_k \{a_k + b_k\}^2 \le 2\sum_k (a_k^2 + b_k^2) \\ &= 2\sum_k \left\{(Q_{s_{k+1}}^{\Delta} - Q_{s_k}^{\Delta})^2 + (Q_{s_{k+1}}^{\Delta'} - Q_{s_k}^{\Delta'})^2\right\} \\ &= 2Q_r^{\Delta\Delta'}(Q^{\Delta}) + 2Q_r^{\Delta\Delta'}(Q^{\Delta'}). \end{aligned}$$

   Thus to show that $EY_r^2 \to 0$ it suffices to show that if $|\Delta| + |\Delta'| \to 0$ then $EQ_r^{\Delta\Delta'}(Q^{\Delta}) \to 0$.

To do this, let $s_k \in \Delta\Delta'$ and $t_j \in \Delta$ so that $t_j \le s_k < s_{k+1} \le t_{j+1}$. From the definition of $Q_s^\Delta = Q_s^\Delta(X)$ we have

$$\begin{aligned} Q_{s_{k+1}}^\Delta - Q_{s_k}^\Delta &= \sum_{t_j \le s_{k+1}} (X_{t_j} - X_{t_{j-1}})^2 - \sum_{t_j \le s_k} (X_{t_j} - X_{t_{j-1}})^2 \\ &= (X_{s_{k+1}} - X_{t_j})^2 - (X_{s_k} - X_{t_j})^2 \equiv (a-b)^2 - (c-b)^2 \\ &= \{(a-b)-(c-b)\}\{(a-b)+(c-b)\} = (a-c)\{a+c-2b\} \\ &= (X_{s_{k+1}} - X_{s_k})(X_{s_{k+1}} + X_{s_k} - 2X_{t_j}). \end{aligned}$$

Squaring and then summing on $s_k \in \Delta\Delta'$ yields, with $j(k) \equiv \sup\{j : t_j \le s_k\}$,

$$\begin{aligned} Q_r^{\Delta\Delta'}(Q^\Delta) &= \sum_{s_k \le r} (X_{s_{k+1}} - X_{s_k})^2 \{X_{s_{k+1}} + X_{s_k} - 2X_{t_{j(k)}}\}^2 \\ &\le \sup_k \{X_{s_{k+1}} + X_{s_k} - 2X_{t_{j(k)}}\}^2 \sum_{s_k \le r} (X_{s_{k+1}} - X_{s_k})^2 \\ &= \sup_k \{X_{s_{k+1}} + X_{s_k} - 2X_{t_{j(k)}}\}^2 Q_r^{\Delta\Delta'}(X). \end{aligned}$$

The the Cauchy-Schwarz inequality yields

$$EQ_r^{\Delta\Delta'}(Q^\Delta) \le \left\{ E \sup_k \{X_{s_{k+1}} + X_{s_k} - 2X_{t_{j(k)}}\}^4 \right\}^{1/2} \cdot \left\{ EQ_r^{\Delta\Delta'}(X)^2 \right\}^{1/2}.$$

The first term on the right side in the last display converges to 0 when $|\Delta| + |\Delta'| \to 0$ by the dominated convergence theorem since X_t is bounded and continuous. Thus it suffices to prove that

$$EQ_r^{\Delta\Delta'}(X)^2 \le 12M^4.$$

2. Durrett, Exercise 3.6, page 52: If X_t is a bounded martingale, then $X_t^2 - \langle X \rangle_t$ is a uniformly integrable martingale.

   Solution: Since $X_t^2 - \langle X \rangle_t$ is a martingale and $|X_t| \le M$ for all t,

   $$E\langle X \rangle_t = EX_t^2 - EX_0^2 \le M^2.$$

   Letting $t \to \infty$ and using monotone convergence it follows that $E\langle X \rangle_\infty \le M^2$. Thus $Y_t \equiv X_t^2 - \langle X \rangle_t$ is dominated by an integrable random variable:

   $$\begin{aligned} |Y_t| &= |X_t^2 - \langle X \rangle_t| \\ &\le |X_t^2| + \langle X \rangle_t \le M^2 + \langle X \rangle_\infty \equiv Y \end{aligned}$$

which is integrable; $EY < \infty$. This implies that $\{Y_t\}$ is uniformly integrable.

3. Durrett, Exercises 3.8 & 3.9, page 52: If $S \le T$ are stopping times and $\langle X\rangle_S = \langle X\rangle_T$, then X is constant on $[S,T]$.
   Conversely, if $S \le T$ are stopping times and X is constant on $[S,T]$, then $\langle X\rangle_S = \langle X\rangle_T$.

   Solution: Let $T_n \equiv \inf\{t : |X_t| > n\}$ and consider the stopped process X^{T_n}. By exercise 3.5 it suffices to prove the result when X is a bounded martingale. By Exercise 3.7 we can suppose without loss that $S = 0$. By the L^2 maximal inequality for the martingale $X_{t\wedge T}$ and then the optional sampling theorem on the martingale $X_t^2 - \langle X\rangle_t$ we find that

$$E\left(\sup_{t\le n} X_{t\wedge T}^2\right) \le 4E(X_{T\wedge n}^2) = E(\langle X\rangle_{T\wedge n}) = 0.$$

   Letting $n \to \infty$ yields $E\left(\sup_{t<\infty} X_{t\wedge T}^2\right) = 0$, and hence X is a.s. constant (namely 0) on $[0,T]$.

4. Durrett, proof of 4.2.c, page 55: Durrett writes "In view of the results in the last paragraph, we can now prove the result by establishing it in the case $H = 1_{(a,b]}C$ and $K = 1_{(c,d]}$, *and* we can furthermore assume that (i) $b \le c$ or (ii) $a = c$, $b = d$." Justify these two claims.

   Solution:

5. Durrett, Exercise 4.2, page 56: $\|H\|_X$ is a norm.

   Solution: We need to show that:
   (i) If $H \in \Pi_2(X)$, and $a \in \mathbb{R}$, then $\|aH\|_X = |a|\|H\|_X$.
   (ii) If $H, K \in \Pi_2(X)$, then $\|H + K\|_X \le \|H\|_X + \|K\|_X$.
   (iii) If $\|H\|_X = 0$, then $H = 0$.
   The proof of (i) is easy since

$$\begin{aligned}\|aH\|_X^2 &= E\int_0^\infty (aH)^2 d\langle X\rangle = E\int_0^\infty a^2H^2 d\langle X\rangle \\ &= a^2E\int_0^\infty H^2 d\langle X\rangle = |a|^2\|H\|_X^2.\end{aligned}$$

To prove (ii), note that

$$\begin{aligned}
\|H+K\|_X^2 &= E\int_0^\infty (H+K)^2 d\langle X\rangle = E\int_0^\infty (H^2+2HK+K^2)d\langle X\rangle \\
&= \|H\|_X^2 + 2E\int_0^\infty HKd\langle X\rangle + \|K\|_X^2 \\
&\le \|H\|_X^2 + 2\Big|E\int_0^\infty HKd\langle X\rangle\Big| + \|K\|_X^2
\end{aligned}$$

where, by the Kunita-Watanabe inequality

$$\begin{aligned}
\Big|\int_0^\infty HKd\langle X\rangle\Big| &= |\int_0^\infty HKd\langle X,X\rangle| \\
&\le \int_0^\infty |HK|d\langle X,X\rangle \\
&\le \left(\int_0^\infty H^2 d\langle X\rangle\right)^{1/2}\left(\int_0^\infty K^2 d\langle X\rangle\right)^{1/2}
\end{aligned}$$

and hence by the Cauchy-Schwarz inequality

$$\begin{aligned}
E|\int_0^\infty HKd\langle X\rangle| &\le E\left\{\left(\int_0^\infty H^2 d\langle X\rangle\right)^{1/2}\left(\int_0^\infty K^2 d\langle X\rangle\right)^{1/2}\right\} \\
&\le \left\{E\int_0^\infty H^2 d\langle X\rangle\right\}^{1/2}\cdot\left\{E\int_0^\infty K^2 d\langle X\rangle\right\}^{1/2} \\
&= \|H\|_X\cdot\|K\|_X.
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
\|H+K\|_X^2 \le \|H\|_X^2 + 2\|H\|_X\|K\|_X + \|K\|_X^2 \\
= (\|H\|_X + \|K\|_X)^2
\end{aligned}$$

and hence (ii) (the triangle inequality) holds.
To prove (iii), suppose that

$$\|H\|_X^2 = E\int_0^\infty H^2 d\langle X\rangle = 0.$$

By the identification of $\|\cdot\|_X$ with the $L_2(\mu)-$norm for the Doleans measure μ on $\Pi_2(X)$ defined by

$$\mu(A\times(s,t]) = E\{1_A\langle X\rangle_t\}$$

for $A \in \mathcal{F}_s$ we have

$$\int H_s^2(\omega))d\mu(\omega, s) = 0,$$

and this implies that $|H_s(\omega)| = 0$ almost everywhere with respect to μ.

The following concerns Durrett's solution to this problem: If we fix ω, then $t \mapsto \langle X \rangle_t$ defines a measure. The triangle inequality for L^2 of that measure implies

$$\left(\int (H_s + K_s)^2 d\langle X \rangle_t\right)^{1/2} \leq \left(\int H_s^2 d\langle X \rangle_t\right)^{1/2} + \left(\int K_s^2 d\langle X \rangle_t\right)^{1/2}.$$

Taking expectations yields

$$E\left\{\left(\int (H_s + K_s)^2 d\langle X \rangle_t\right)^{1/2}\right\} \leq E\left\{\left(\int H_s^2 d\langle X \rangle_t\right)^{1/2}\right\} + E\left\{\left(\int K_s^2 d\langle X \rangle_t\right)^{1/2}\right\}.$$

This is not quite the inequality claimed in the Exercise as stated since

$$E\left\{\left(\int H_s^2 d\langle X \rangle_t\right)^{1/2}\right\} \neq \left\{E\left(\int H_s^2 d\langle X \rangle_t\right)\right\}^{1/2} \equiv \|H\|_X.$$

Since $EY^{1/2} \leq (EY)^{1/2}$ by concavity of $\varphi(x) = x^{1/2}$ we do have

$$E\left\{\left(\int (H_s + K_s)^2 d\langle X \rangle_t\right)^{1/2}\right\} \leq \left\{E\left(\int H_s^2 d\langle X \rangle_t\right)\right\}^{1/2} + \left\{E\left(\int K_s^2 d\langle X \rangle_t\right)\right\}^{1/2},$$

but this does not yield the triangle inequality either.

An alternative solution is via the Doléans measure μ defined above: for $A \in \mathcal{F}_s$,

$$\mu(A \times (s, t]) = E\{1_A \langle X \rangle_t\}$$

Then $\|H\|_X$ is indeed the $L_2(\mu)$ norm on the space Π of predictable functions:

$$\|H\|_X \equiv \left\{E\left(\int H_s^2 d\langle X \rangle_t\right)\right\}^{1/2} = \left(\int H_s^2(\omega))d\mu(\omega, s)\right)^{1/2}$$

and hence the triangle inequality for $L_2(\mu)$ holds. In this sense $\|\cdot\|_X$ is not a norm, but a pseudo-norm (or semi-norm).

6. Durrett, Exercise 4.3, page 56: $X \in \mathcal{M}^2$ if and only if $E(X_0^2) < \infty$ and $E\langle X\rangle_\infty < \infty$.

   Solution: If $X \in \mathcal{M}^2$, then by definition $(\sup_t E(X_t^2))^{1/2} < \infty$ and hence both $EX_0^2 < \infty$ and $E(\sup_t X_t^2) < \infty$ by Doob's L^2-maximal inequality. To see that $E\langle X\rangle_\infty < \infty$, let T_n be a sequence of stopping times $\uparrow \infty$ so that X^{T_n} is bounded and $\langle X\rangle_{T_n} \le n$. Then by optional sampling

   $$E(X_{T_n}^2 - \langle X\rangle_{T_n})1_{[T_n>0]} = EX_0^2 1_{[T_n>0]}.$$

   Letting $n \to \infty$ yields $E(X_\infty^2 - \langle X\rangle_\infty) = EX_0^2$, and hence $E\langle X\rangle_\infty = E(X_\infty^2) - EX_0^2 < \infty$.

   To prove the converse, note that since X is a local martingale (in order to define $\langle X\rangle$), and optional stopping for the stopping times T_n yields the previous display. Rearranging this we find that

   $$\begin{aligned} E(X_{T_n}^2 1_{[T_n>0]}) &= EX_0^2 1_{[T_n>0]} + E\langle X\rangle_{T_n} 1_{[T_n>0]} \\ &\le EX_0^2 + E\langle X\rangle_\infty < \infty \end{aligned}$$

   and hence X is L^2-bounded.

7. Durrett, Exercise 4.5, page 59: If X is a bounded martingale and $H \in \Pi_2(X)$, then $\|H \cdot X\|_2 = \|H\|_X$.

   Solution: Let $H^n \in b\Pi_1$ with $\|H^n - H\|_X \to 0$. Since $\|(H^n \cdot X) - (H \cdot X)\|_2 \to 0$ it follows from Exercise 4.4, the isometry property for $b\Pi_1$ and Exercise 4.4 again that

   $$\|H \cdot X\|_2 = \lim_n \|H^n \cdot X\|_2 = \lim_n \|H^n\|_X = \|H\|_X.$$

   (Exercise 4.4 is as follows: if $\|\cdot\|$ is a norm and $\|x_n - x\|_2 \to 0$ then $\|x_n\| \to \|x\|$.
   This follows easily since $\|x\| \le \|x_n\| + \|x_n - x\|$ by the triangle inequality and hence $\|x\| \le \liminf_n \|x_n\|$. On the other hand $\|x_n\| \le \|x\| + \|x_n - x\|$ by the triangle inequality again, so $\limsup_n \|x_n\| \le \|x\|$. Combining these yields

   $$\|x\| \le \liminf_n \|x_n\| \le \limsup_n \|x_n\| \le \|x\|,$$

   and hence $\lim_n \|x_n\| = \|x\|$.)

8. Durrett, Theorem 6.5, page 65: If X, Y are continuous local martingales, $H \in \Pi_3(X)$ and $K \in \Pi_3(Y)$, then

$$\langle H \cdot X, K \cdot Y \rangle_t = \int_0^t H_s K_s d\langle X, Y \rangle_s.$$

Prove this theorem.

Solution: Let

$$T_n \equiv \inf\{t > 0 : \ |X|_t \vee \int_0^t H_s^2 d\langle X \rangle_s \vee |Y_t| \vee \int_0^t K_s^2 d\langle Y \rangle_s > n\}.$$

Then by stopping at T_n it suffices to prove that the claimed equality holds for bounded martingales X and Y, $H \in \Pi_2(X)$, $K \in \Pi_2(Y)$. But this holds by Theorem 5.4.

9. Durrett, Exercise 6.2, page 66: If X is a continuous local martingale and $H \in \Pi_2(X)$, then $H \cdot X \in \mathcal{M}^2$ and $\|H \cdot X\|_2 = \|H\|_X$. (Here I am not sure I believe the claim: we start with a local martingale X and end up with a martingale $H \cdot X$, at least according to Durrett's claim. Thus you may need to rephrase the claim slightly. Alternatively, give a counter-example.)

Solution: Let

$$T_n \equiv \inf\{t > 0 : \ |X_t| \vee \int_0^t H_s^2 d\langle X \rangle_s > n\},$$

and consider the process X stopped at T_n; i.e. X^{T_n}. Since X^{T_n} is a bounded martingale, it follows from Problem 7 (Exercise 4.5) and Lemma 3.7 that

$$\begin{aligned} \|H \cdot X^{T_n}\|_2^2 &= \|H\|_{X^{T_n}}^2 = E \int H_2^2 d\langle X^{T_n} \rangle_s = E \int H_s^2 d\langle X \rangle_s^{T_n} \quad (1) \\ &= E \int_0^{T_n} H_s^2 d\langle X \rangle_s \le E \int H_s^2 d\langle X \rangle_s < \infty \quad (2) \end{aligned}$$

since $H \in \Pi_2(X)$ and since $\int_0^{T_n} H_s^2 d\langle X \rangle_s \le \int_0^\infty H_s^2 d\langle X \rangle_s$. By Doob's L^2-maximal inequality we find that

$$E\left(\sup_{t \le T_n} (H \cdot X)_t^2\right) \le 4E \int_0^{T_n} H_s^2 d\langle X \rangle_s \le 4E \int_0^\infty H_s^2 d\langle X \rangle_s < \infty.$$

Letting $n \to \infty$ and using monotone convergence yields

$$E\left(\sup_t (H \cdot X)_t^2\right) < \infty,$$

and hence $H \cdot X \in \mathcal{M}^2$. This provides an integrable dominating function for application of the dominated convergence theorem in (**??**): letting $n \to \infty$ on both sides of (**??**) yields $\|H \cdot X\|_2 = \|H\|_X$.

10. Durrett, Exercise 6.3, page 66: Let X be a continuous local martingale. Let $S \le T < \infty$ be stopping times, let $C(\omega)$ be bounded with $C(\omega) \in \mathcal{F}_S$, and define $H_s = C1_{(S,T]}(s)$. Then $H \in \Pi_3(X)$ and

$$\int H_s dX_s = C(X_T - X_S).$$

Solution: By stopping we can reduce to the case in which X is a bounded continuous martingale and $\langle X \rangle_t \le N$ for all t, which implies $H \in \Pi_2(X)$. Now replace S and T by S_n and T_n which stop at the next dyadic rational and let $H_s^n = C$ for $S_n < s \le T_n$ then $H_s^n \in \Pi_1$ and it follows from the definition of the integral in Step 2 (Section 2.4) that

$$\int H_s^n dX_s = C(X_{T_n} - X_{S_n}).$$

But if $|C| \le K$ a.s. then

$$\begin{aligned} \|H^n - H\|_X &\le K^2\{E(\langle X \rangle_{T_n} - \langle X \rangle_T) + E(\langle X \rangle_{T_n} - \langle X \rangle_T)\} \\ &\to 0 \end{aligned}$$

by the dominated convergence theorem. Using Exercise 4.4 and (4.3.b) it follows that $\|(H^n \cdot X) - (H \cdot X)\|_2 \to 0$, and hence $\int H_s^n dX_s \to_p \int H_s dX_s$. Clearly $C(X_{T_n} - X_{S_n}) \to_{a.s.} C(X_T - X_S)$, and hence the claimed equality holds.

11. Durrett, Exercise 6.4, page 67: If X is a continuous local martingale, then

$$\int_0^t 2X_s dX_s = X_t^2 - X_0^2 - \langle X \rangle_t.$$

Solution: Note that for a partition $\{t_i^n : 0 \le i \le k_n\}$ with $t_0^n \equiv 0$ and $t_{k_n}^n = t$ we have

$$\begin{aligned} X_t^2 - X_0^2 &= \sum_{i=0}^{k_n-1}(X_{t_{i+1}^n}^2 - X_{t_i^n}^2) \\ &= \sum_{i=0}^{k_n-1}(X_{t_{i+1}^n} - X_{t_i^n})^2 + \sum_{i=0}^{k_n-1} 2X_{t_i^n}\{X_{t_{i+1}^n} - X_{t_i^n}\} \end{aligned}$$

since $b^2 - a^2 = (b-a)(b+a) = (b-a)(b-a+2a) = (b-a)^2 + 2a(b-a)$. By Theorem 3.8 the first term converges to $\langle X \rangle_t$. By Theorem 6.7 the second term converges to $2\int_0^t X_s dX_s$. Thus

$$X_t^2 - X_0^2 = 2\int_0^t X_s dX_s + \langle X \rangle_t,$$

and the claimed equality holds. Alternatively, from Itô's formula with $f(x) = x^2$, we have $f'(x) = 2x$ and $f''(x) = 2$, and hence

$$\begin{aligned} X_t^2 - X_0^2 &= f(X_t) - f(X_0) \\ &= \int_0^2 f'(X_s)dX_s + (1/2)\int_0^t f''(X_s)d\langle X \rangle_s \\ &= \int_0^t 2X_s dX_s + (1/2)\int_0^t 2d\langle X \rangle_s \\ &= \int_0^t 2X_s dX_s + \langle X \rangle_t, \end{aligned}$$

and rearranging yields the claimed identity.

12. Durrett, Exercise 6.5, page 67: Show that if X is a continuous local martingale and we evaluate at the right end point then

$$\sum_i 2X_{t_{i+1}^n}\{X(t_{i+1}^n) - X(t_i^n)\} \to_p \int_0^t 2X_s dX_s + 2\langle X \rangle_t = X_t^2 - X_0^2 + \langle X \rangle_t.$$

Solution: Using the same notation as in the previous problem we have, using $2b(b-a) - 2a(b-a) = 2(b-a)^2$

$$\begin{aligned} & \sum_{i=0}^{k_n-1} 2X_{t_{i+1}^n}\{X_{t_{i+1}^n} - X_{t_i^n}\} - \sum_{i=0}^{k_n-1} 2X_{t_i^n}\{X_{t_{i+1}^n} - X_{t_i^n}\} \\ &= \sum_{i=0}^{k_n-1} (X_{t_{i+1}^n} - X_{t_i^n})^2 \to_p 2\langle X \rangle_t. \end{aligned}$$

Thus the previous problem yields

$$\begin{aligned} & \sum_i 2X_{t_{i+1}^n}\{X(t_{i+1}^n) - X(t_i^n)\} \\ &= \sum_{i=0}^{k_n-1} 2X_{t_{i+1}^n}\{X_{t_{i+1}^n} - X_{t_i^n}\} - \sum_{i=0}^{k_n-1} 2X_{t_i^n}\{X_{t_{i+1}^n} - X_{t_i^n}\} \\ &\quad + \sum_{i=0}^{k_n-1} 2X_{t_i^n}\{X_{t_{i+1}^n} - X_{t_i^n}\} \\ &= \sum_{i=0}^{k_n-1} 2X_{t_{i+1}^n}\{X_{t_{i+1}^n} - X_{t_i^n}\} - \sum_{i=0}^{k_n-1} 2X_{t_i^n}\{X_{t_{i+1}^n} - X_{t_i^n}\} \\ &\quad + X_t^2 - X_0^2 - \sum_{i=0}^{k_n-1} (X_{t_{i+1}^n} - X_{t_i^n})^2 \\ &\to_p 2\langle X \rangle_t + X_t^2 - X_0^2 - \langle X \rangle_t \\ &= X_t^2 - X_0^2 + \langle X \rangle_t = 2\int_0^t X_s dX_s + 2\langle X \rangle_t. \end{aligned}$$

13. Let $f \in L_2[(0,\infty), \lambda]$ and consider the process $Z(t) \equiv \exp(\int_0^t f(s)dB(s) - \frac{1}{2}\int_0^t f^2(s)ds)$ where B is standard Brownian motion. (a) Compute $E\{Z(t)|\mathcal{F}_s\}$ where $\mathcal{F}_s$ is the $\sigma-$field generated by $\{B_r : \ r \le s\}$. Is $\{Z_t\}$ a (local-) martingale?
(b) When $f(t) = \mu \ne 0$, the process $Z(t) = \exp(\mu B(t) - (1/2)\mu^2 t)$ yields the Radon-Nikodym derivative $dP_{\mu,t}/dP_{0,t}$ of $P_\mu|_{\mathcal{C}_t}$ with respect to $P_0|_{\mathcal{C}_t}$ where P_μ is the law of Brownian motion with drift, $B_\mu(t) = B(t) + \mu t$, on $(C[0,\infty), \mathcal{C}_{[0,\infty)})$; recall problem 4 of Problem set 6. Does Z in part (a) have a similar interpretation for a general L_2 function f? [See Durrett, Theorem 3.7 and Section 5.5.]

Solution: (a) We compute

$$
\begin{aligned}
&E(Z_t|\mathcal{F}_s)\\
&= E\left\{\exp\left(\int_0^t f(r)dB_r\right)\big|\mathcal{F}_s\right\}\cdot\exp\left(-(1/2)\int_0^t f^2(r)dr\right)\\
&= E\left\{\exp\left(\int_0^s f(r)dB_r\right)\cdot\exp\left(\int_s^t f(r)dB_r\right)\big|\mathcal{F}_s\right\}\cdot\exp\left(-(1/2)\int_0^t f^2(r)dr\right)\\
&= \exp\left(\int_0^s f(r)dB_r\right)E\left\{\exp\left(\int_s^t f(r)dB_r\right)\right\}\cdot\exp\left(-(1/2)\int_0^t f^2(r)dr\right)
\end{aligned}
$$

almost surely since $\int_0^s f(r)dB_r \in \mathcal{F}_s$ and $\int_s^t f(r)dB_r = \int_s^t f(r)d(B_r - B_s)$ is independent of $\mathcal{F}_s$. But we also recall that $\int_s^r f(r)dB(r) \sim N(0, \int_s^t f^2(r)dr)$, and hence

$$
E\exp\left(\int_s^t f(r)dB_r\right) = \exp\left(\frac{1}{2}\int_s^t f^2(r)dr\right).
$$

Putting this together yields

$$
E(Z_t|\mathcal{F}_s) =_{a.s.} \exp\left(\int_0^s f(r)dB_r - (1/2)\int_0^s f^2(r)dr\right) = Z_s,
$$

and hence $\{Z_t, \mathcal{F}_t\}$ is a martingale.

(b) In Durrett's Theorem 12.4 we take $\alpha_t = Z_t$ and take $X_t = B_t$ under P. Then I claim that

$$
\langle Z, X\rangle_t \;=\; \langle Z, B\rangle_t = \int_0^t Z_s f(s)d\langle B\rangle_s = \int_0^t Z_s f(s)ds.
$$

This follows from Theorem 8.7 and the calculation in class on 5 June showing that $Z_t = \int_0^t Z_s f(s)dX_s$. Then Theorem 8.7 yields

$$
\begin{aligned}
\langle Z, X\rangle_t &= \langle\int_0^\cdot Z_s f(s)dX_s, X\rangle_t = \langle\int_0^\cdot Z_s f(s)dB_s, B\rangle_t\\
&= \int_0^t Z_s f(s)d\langle B\rangle_s = \int_0^t Z_s f(s)ds.
\end{aligned}
$$

Then from the Girsanov formula of (12.4),

$$
\begin{aligned}
A_t &= \int_0^t \frac{1}{Z_s}d\langle Z, X\rangle_s = \int_0^t \frac{1}{Z_s}d\left(\int_0^\cdot Z_r f(r)dr\right)\\
&= \int_0^t \frac{1}{Z_s}Z_s f(s)ds = \int_0^t f(s)ds.
\end{aligned}
$$

Thus under Q, $X_t - A_t = X_t - \int_0^t f(s)ds (= B_t)$ is a martingale.

14. (Sources for harmonic functions) A fact from complex variables is that the real and imaginary parts of an analytic function satisfy the Cauchy-Riemann equations; that is, if $f(z)$ is a differentiable function of $z = x + iy$ and if we write $f(x+iy) = u(x,y) + iv(x,y)$, then

$$\partial u/\partial x = \partial v/\partial y \quad \text{and} \quad \partial u/\partial y = -\partial v/\partial x.$$

(a) Use the equations in the last display to show that $u(x,y)$ and $v(x,y)$ are harmonic. What harmonic functions do you obtain from the real and imaginary parts of the analytic functions e^z and ze^z?
(b) Consider the harmonic functions f you found in (a). What is the result of applying Ito's formula to the processes of the form $X_t = f(B_t)$ where $\underline{B}_t$ is 2-dimensional Brownian motion?
(c) Consider the family of hyperbolas given by $H_\alpha = \{(x,y) : x^2 - y^2 = \alpha\}$. What is the probability that the standard two-dimensional Brownian motion in $\mathbb{R}^2$ starting at $(2,0)$ will hit $H(1)$ before hitting $H(5)$? Hint: consider the harmonic functions obtained from the complex function $f(z) = z^2$.

Solution: (a) Now

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x},$$

so

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

since the two mixed partial derivatives in the last line are equal. Similarly,

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}, \quad \text{and} \quad \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x},$$

so

$$\triangle v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} = 0.$$

For $f(z) = e^z = e^{x+iy} = e^x(\cos y + i \sin y) = e^x \cos y + i e^x \sin y$ we have $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$.
For

$$\begin{aligned} f(z) &= ze^z = (x+iy)e^{x+iy} = (x+iy)e^x(\cos x + i \sin y) \\ &= e^x(x \cos y - y \sin y) + ie^x(y \cos y + x \sin y) \end{aligned}$$

we have $u(x,y) = e^x(x\cos y - y\sin y)$ and $v(x,y) = e^x(y\cos y + x\sin y)$.
(b) If f is harmonic, then for $\underline{B}_t$ = 2-dimensional Brownian motion started at $\underline{x}_0$, Itô's formula yields

$$\begin{aligned} f(\underline{B}_t) - f(\underline{B}_0) &= \int_0^t \nabla f(\underline{B}_s) \cdot d\underline{B}_s + \frac{1}{2}\int_0^t \triangle f(\underline{B}_s) ds \\ &= \int_0^t \nabla f(\underline{B}_s) \cdot d\underline{B}_s. \end{aligned}$$

Thus $f(\underline{B}_t)$ is a martingale.
(c) When $f(z) = z^2 = (x+iy)^2 = (x^2 - y^2) + i2xy$ we have $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. Thus $u(\underline{B}_t)$ is a martingale. To emphasize that $\underline{B}$ may start at $\underline{x}_0 \neq 0$ we write $\underline{X}_t \equiv \underline{B}_t + \underline{x}_0$ where $\underline{B}_0 = \underline{0}$. Now let $H_\alpha \equiv \{(x,y) : \ x^2 - y^2 = \alpha\}$, and define

$$\tau_\alpha \equiv \inf\{t > 0 : \ \underline{X}_t \in H_\alpha\},$$

and for $\alpha < \beta$ let $\tau_{\alpha,\beta} \equiv \tau_\alpha \wedge \tau_\beta$. Thus $\{\tau_{\alpha,\beta} = \tau_\alpha\} = \{\tau_\alpha < \tau_\beta\}$ is the event that 2-dimensional Brownian motion started from $\underline{x}_0$ hits H_α before it hits H_β. By optional sampling with $\underline{x}_0$ on the $x-$axis and with $x_{0,1} \in (\alpha^2, \beta^2)$ we have

$$\begin{aligned} u(\underline{x}_0) &= Eu(X_{\tau_{\alpha,\beta}}) \\ &= u(X_{\tau_\alpha})P(\tau_\alpha < \tau_\beta) + u(X_{\tau_\beta})P(\tau_\beta < \tau_\alpha) \\ &= \alpha p + \beta(1-p) = \beta - (\beta - \alpha)p \end{aligned}$$

where $p \equiv P(\tau_\alpha < \tau_\beta)$. It follows that

$$p = P(\tau_\alpha < \tau_\beta) = \frac{\beta - u(\underline{x}_0)}{\beta - \alpha}.$$

When $\beta = 5$, $\alpha = 1$ and $\underline{x}_0 = (2,0)$ we find that

$$p = \frac{5 - (2^2 - 0^2)}{5 - 1} = \frac{5-4}{4} = \frac{1}{4}.$$