Statistics 523, Problem Set 7 Solutions

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1. Let $X(t)$ be a continuous local martingale and let φ be a convex function. Show that $\varphi(X_t)$ is a local submartingale.

   Solution: Let $T_n \equiv \inf\{t : \ |X_t| > n\}$. By Durrett (2.4), page xx, this sequence will reduce X_t. Note that $X_t^{T_n} \le n$. Jensen's inequality for conditional expectation (see e.g. (1.1.d) implies

   $$E\{\varphi(X_t^{T_n})|\mathcal{F}_{T_n \wedge s}\} \ge \varphi(E(X_t^{T_n}|\mathcal{F}_{T_n\wedge s})) = \varphi(X_s^{T_n}).$$

   Thus $\{\varphi(X_t), \mathcal{F}_t\}$ is a local submartingale.

2. Let X_t be a continuous local martingale and let $R < \infty$ be a stopping time. Show that $Y_s \equiv X_{R+s}$ is a local martingale with respect to $\mathcal{G}_s \equiv \mathcal{F}_{R+s}$.

   Solution: Let $T_n \le n$ be a sequence which reduces X, and let $S_n = (T_n - R)^+$. If $s < t$, then it follows from the definitions, the optional sampling theorem, and the fact that $1\{T_n > R\} \in \mathcal{F}_R \subset \mathcal{F}_{R+s}$, that

   $$\begin{aligned} E(Y_{t\wedge S_n} 1_{[S_n>0]}|\mathcal{G}_s) &= E(X_{(R+t)\wedge T_n} 1_{[T_n>r]}|\mathcal{F}_{R+s}) \\ &=_{a.s.} 1_{[T_n>R]} E(X_{(R+t)\wedge T_n}|\mathcal{F}_{R+s}) \\ &=_{a.s.} X_{(R+s)\wedge T_n} 1_{[T_n>R]} = Y_{s\wedge S_n} 1_{[S_n>0]}. \end{aligned}$$

   Thus $\{Y_s, \mathcal{G}_s\}$ is a local martingale.

   Revised solution: Let $Y_s \equiv X_{R+s}$ and $\mathcal{G}_s \equiv \mathcal{F}_{R+s}$. Let $T_n \le n$ be a sequence of stopping times which reduces X. Let $S_n \equiv (T_n - R)^+$. Then we want to show that

   $$E(Y_t^{S_n}|\mathcal{G}_{s\wedge S_n}) =_{a.s.} Y_s^{S_n}, \qquad (1)$$

that is, the sequence $\{S_n\}$ reduces Y. But we compute

$$\begin{aligned}
E(Y_t^{S_n}|\mathcal{G}_{s\wedge S_n}) &= E(Y_{t\wedge S_n}1_{[S_n>0]}|\mathcal{G}_{s\wedge S_n}) \\
&= E\{X_{R+(t\wedge S_n)}1_{[T_n>R]}|\mathcal{F}_{R+(s\wedge S_n)}\} \\
&= E\{X_{(R+t)\wedge T_n}1_{[T_n>R]}|\mathcal{F}_{(R+s)\wedge T_n}\} \\
&= 1_{[T_n>R]}EE\{X_{(R+t)\wedge T_n}|\mathcal{F}_{(R+s)\wedge T_n}\} \\
&\qquad \text{since } [T_n>R]\in\mathcal{F}_R\subset\mathcal{F}_{R+s} \text{ and } [T_n>R]\in\mathcal{F}_{T_n} \\
&=_{a.s.} 1_{[S_n>0]}X_{(R+s)\wedge T_n} \\
&= Y_s^{S_n}.
\end{aligned}$$

where the next to last equality holds by optional sampling: stopping $X_t^{S_n}$ at the two stopping times $R+s \le R+t$ yields the (two-term) martingale $\{X_{R+s}^{T_n}, X_{R+t}^{T_n}\}$. Thus (1) holds; i.e. Y is a local martingale.

3. Show that if X is a continuous local martingale with $X_t \ge 0$ and $EX_0 < \infty$, then X_t is a supermartingale.

   Solution: Let T_n be a sequence of stopping times that reduces X. By the martingale property of the reduced sequence it follows that for $A \in \mathcal{F}_0$ we have

   $$E(1_A X_{t\wedge T_n}1_{[T_n>0]}) = E(1_A X_0 1_{[T_n>0]}).$$

   By Fatou's lemma and the dominated convergence theorem this yields

   $$\begin{aligned}
E(1_A X_t) &\le \liminf_{n\to\infty} E(1_A X_{t\wedge T_n}1_{[T_n>0]}) \\
&\le \lim_{n\to\infty} E(1_A X_0 1_{[T_n>0]}) = E(1_A X_0).
\end{aligned}$$

   By taking $A = \Omega$ this yields $E(X_t) \le EX_0 < \infty$. Since the inequality holds for all $A \in \mathcal{F}_0$ we can conclude that $E(X_t|\mathcal{F}_0) \le X_0$ almost surely. To see that $E(X_t|\mathcal{F}_s) \le X_s$ almost surely for $0 \le s < t$, apply the previous argument to $Y_t = X_{s+t}$ which is a local martingale by the previous problem.

4. Definition: If X and Y are two continuous local martingales, define

   $$\langle X, Y\rangle_t \equiv \frac{1}{4}(\langle X+Y\rangle_t - \langle X-Y\rangle_t)$$

where $\langle X\rangle_t$ is the unique continuous increasing process that makes $X_t^2 - \langle X, Y\rangle_t$ a local martingale. Show that:
(i) $\langle X+Y, Z\rangle_t = \langle X, Z\rangle_t + \langle Y, Z\rangle_t$;
(ii) $\langle X - X_0, Z\rangle_t = \langle X, Z\rangle_t$;
(iii) If $a, b \in \mathbb{R}$, then $\langle aX, bY\rangle_t = ab\langle X, Y\rangle_t$;
(iv) If $a \in \mathbb{R}$, then $\langle aX\rangle_t = a^2\langle X, \rangle_t$.

Solution: (i) Note that

$$\begin{aligned}(X+Y)_t Z_t - (\langle X, Z\rangle_t + \langle Y, Z\rangle_t) &= X_t Z_t + Y_t Z_t - \langle X, Z\rangle_t - \langle Y, Z\rangle_t \\ &= (X_t Z_t - \langle X, Z\rangle_t) + (Y_t Z_t - \langle Y, Z\rangle_t)\end{aligned}$$

is the sum of two local martingales and hence is a local martingale. By a.s. uniqueness of the covariation process, this implies that (i) holds.
(ii) Now $-X_0 Z_t$ is a local martingale, so if $Y_t \equiv X_0$, it follows that $\langle Y, Z\rangle_t \equiv 0$, and then (ii) follows from (i).
(iii) For $a, b \in \mathbb{R}$ the process

$$(aX_t)(bY_t) - ab\langle X, Y\rangle_t = ab(X_t Y_t - \langle X, Y\rangle_t)$$

is a local martingale, and hence (iii) holds.
(iv) This follows immediately from (iii) by taking $a = b$ and $Y = X$.