Statistics 523, Problem Set 6 Solutions

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1. Let $\mathbb{U}_n = \sqrt{n}(\mathbb{G}_n - I)$ be the uniform empirical process based on $\xi_1, \ldots, \xi_n$ i.i.d. Uniform$(0,1)$ random variables. Consider the functional $g(x) = \int_0^1 x(t)dt$.
(a) What is the limiting distribution of $g(\mathbb{U}_n)$?
(b) Compute $g(\mathbb{U}_n)$ explicitly in terms of the ξ_i's.
(c) Use a standard result to find the limiting distribution of $g(\mathbb{U}_n)$ as computed in (b) in a different way. Does this result agree with what you found in (a)?

   Solution: (a) Since g is continuous with respect to the uniform metric on $D[0,1]$ we have $g(\mathbb{U}_n) = \int_0^1 \mathbb{U}_n(t)dt \rightarrow_d \int_0^1 \mathbb{U}(t)dt$. Since this is a linear function of a Gaussian process with finite variance given by

$$\begin{aligned} E\left(\int_0^1 \mathbb{U}(s)ds\right)^2 &= E\int_0^1 \mathbb{U}(s)ds \int_0^1 \mathbb{U}(t)dt = \int_0^1\int_0^1 E\{\mathbb{U}(s)\mathbb{U}(t)\}dsdt \\ &= \int_0^1\int_0^1 (s\wedge t - st)dsdt \\ &= 2\int_0^1 \left(\int_0^t sds\right)dt - \left(\int_0^1 sds\right)^2 \\ &= \int_0^1 t^2 dt - (1/2)^2 = (1/3) - (1/4) = 1/(12), \end{aligned}$$

   it follows that the limiting distribution is $N(0, 1/12)$.
(b) Now

$$\begin{aligned} g(\mathbb{U}_n) &= \int_0^1 \sqrt{n}(\mathbb{G}_n(t) - t)dt = -\sqrt{n}\left(\int_0^1 (1-\mathbb{G}_n(t))dt - \int_0^1 (1-t)dt\right) \\ &= -\sqrt{n}(\overline{\xi}_n - 1/2). \end{aligned}$$

   (c) By (b) the the central limit theorem we have

$$\sqrt{n}(\overline{\xi}_n - 1/2) \rightarrow_d N(0, \mathrm{Var}(\xi_1)) = N(0, 1/12),$$

   so that $g(\mathbb{U}_n) = -\sqrt{n}(\overline{\xi}_n - 1/2) \rightarrow_d -N(0,1/12) = N(0,1/12)$ exactly as in (a).

2. Consider the uniform empirical process $\mathbb{U}_n$ as in problem 1 above and let
$$g(x) = \int_0^1 \frac{x^2(t)}{t(1-t)} dt.$$
Show that $g(\mathbb{U}_n) \to_d g(\mathbb{U})$. [Note that g is not continuous with respect to $\| \cdot \|_\infty \equiv \| \cdot \|$, the uniform metric on D. Hint: for $0 < \delta < 1/2$, consider the intervals $(0, \delta]$ and $[1-\delta, 1)$ separately.]

Solution: Let $\delta > 0$. Note that $E\mathbb{U}_n(t)^2 = Var(\mathbb{U}_n(t)) = t(1-t) = Var(\mathbb{U}(t)) = E\mathbb{U}^2(t)$ for each $t \in [0,1]$. Hence by Markov's inequality and Fubini's theorem, for $\epsilon > 0$ we have
$$P\left(\int_0^\delta \frac{\mathbb{U}_n^2(t)}{t(1-t)} dt > \epsilon\right) \leq \epsilon^{-1} \int_0^\delta \frac{E\mathbb{U}_n^2(t)}{t(1-t)} dt = \epsilon^{-1} \int_0^\delta 1\, dt = \epsilon^{-1}\delta.$$
Choosing $\delta = \epsilon^2$ yields
$$P\left(\int_0^{\epsilon^2} \frac{\mathbb{U}_n^2(t)}{t(1-t)} dt > \epsilon\right) < \epsilon$$
for every $n \geq 1$. This same argument yields
$$\begin{aligned}
&P\left(\int_{1-\epsilon^2}^1 \frac{\mathbb{U}_n^2(t)}{t(1-t)} dt > \epsilon\right) < \epsilon, \quad \text{for all} \quad n \geq 1, \\
&P\left(\int_0^{\epsilon^2} \frac{\mathbb{U}^2(t)}{t(1-t)} dt > \epsilon\right) < \epsilon, \quad \text{and} \\
&P\left(\int_{1-\epsilon^2}^1 \frac{\mathbb{U}^2(t)}{t(1-t)} dt > \epsilon\right) < \epsilon.
\end{aligned}$$
Thus we find, for the Skorokhod constructed versions of $\mathbb{U}_n$ with $\|\mathbb{U}_n - \mathbb{U}\| \to_p 0$,
$$\begin{aligned}
\left|g(\mathbb{U}_n) - g(\mathbb{U})\right| &\leq \int_0^{\epsilon^2} \frac{\mathbb{U}_n^2(t)}{t(1-t)} dt + \int_0^{\epsilon^2} \frac{\mathbb{U}^2(t)}{t(1-t)} dt \\
&\quad + \int_{1-\epsilon^2}^1 \frac{\mathbb{U}_n^2(t)}{t(1-t)} dt + \int_{1-\epsilon^2}^1 \frac{\mathbb{U}^2(t)}{t(1-t)} dt \\
&\quad + \|\mathbb{U}_n - \mathbb{U}\|_\delta^{1-\delta} \int_{\epsilon^2}^{1-\epsilon^2} \frac{1}{t(1-t)} dt \\
&\leq 4\epsilon + C_\epsilon \|\mathbb{U}_n - \mathbb{U}\| \\
&\to_p 4\epsilon
\end{aligned}$$

as $n \to \infty$ since $\|\mathbb{U}_n - \mathbb{U}\| \to_p 0$, and the set on which this occurs has probability $\geq 1 - 4\epsilon$. It follows that $g(\mathbb{U}_n) \to_p g(\mathbb{U})$ for the special construction, and hence $g(\mathbb{U}_n) \to_d g(\mathbb{U})$ for an version of the processes $\mathbb{U}_n$.

3. Use a reflection principle to show that for $0 \leq y \leq x$

$$P(\sup_{0 \leq s \leq t} \mathbb{S}(s) \geq x,\ \mathbb{S}(t) \leq y) = P(\mathbb{S}(t) \geq 2x - y)\,,$$

and use this to show that the joint density of $M^+ \equiv \sup_{0 \leq s \leq t} \mathbb{S}(s), \mathbb{S}(t)$ is given by

$$f(x,y) = \sqrt{\frac{2}{\pi t^3}}(2x - y)\exp\left(-\frac{(2x-y)^2}{2t}\right) \qquad \text{for } \ 0 \leq y \leq x\,.$$

Solution: Note that by reflection

$$\begin{aligned} P(\sup_{0 \leq s \leq t} \mathbb{S}(s) \geq x,\ \mathbb{S}(t) \leq y) &= P(\tau_x \leq t, \mathbb{S}(\tau_x + t - \tau_x) - \mathbb{S}(\tau_x) \leq y - x) \\ &= P(\tau_x \leq t, \mathbb{S}(\tau_x + t - \tau_x) - \mathbb{S}(\tau_x) \geq x - y) \\ &= P(\tau_x \leq t, \mathbb{S}(t) \geq 2x - y) \\ &= P(\mathbb{S}(t) \geq 2x - y) \\ &= 1 - \Phi\left(\frac{2x - y}{\sqrt{t}}\right) \end{aligned}$$

since $[\mathbb{S}(t) \geq 2x - y] \subset [\tau_x \leq t]$. Thus we compute the joint density f of $M^+ \equiv \sup_{0 \leq s \leq t} \mathbb{S}(s), \mathbb{S}(t)$ as

$$\begin{aligned} f(x,y) &= -\frac{\partial^2}{\partial x \partial y} P(\sup_{0 \leq s \leq t} \mathbb{S}(s) \geq x,\ \mathbb{S}(t) \leq y) \\ &= \frac{\partial}{\partial y}\phi\left(\frac{2x-y}{\sqrt{t}}\right)\frac{2}{\sqrt{t}} \\ &= \phi'\left(\frac{2x-y}{\sqrt{t}}\right)\frac{-1}{\sqrt{t}}\frac{2}{\sqrt{t}} \\ &= \frac{2x-y}{\sqrt{t}}\phi\left(\frac{2x-y}{\sqrt{t}}\right)\frac{2}{t} \\ &= \sqrt{\frac{2}{\pi t^3}}(2x-y)\exp\left(-\frac{(2x-y)^2}{2t}\right) \qquad \text{for } \ 0 \leq y \leq x \end{aligned}$$

where we used $\phi'(x) = -x\phi(x)$ in the next to last equality.

4. Suppose that $\mathbb{S}$ is standard Brownian motion on $(C[0,\infty), \mathcal{C}_{[0,\infty)})$, and let its distribution be denoted by $P = P_0$. Let $\mathbb{S}_\mu(t) \equiv \mathbb{S}(t) + \mu t$ be Brownian motion with drift μ, and let P_μ denote the distribution of $\mathbb{S}_\mu$ on $(C[0,\infty), \mathcal{C}_{[0,\infty)})$. Set $Y(t) \equiv \exp(\mu\mathbb{S}(t) - \mu^2 t/2)$. For $t > 0$ let $P_{0,t}$ and $P_{\mu,t}$ denote the distributions P_0 and P_μ restricted to $\mathcal{A}_t \equiv \{\mathbb{S}(s) : s \le t\}$. Show that the Radon - Nikodym derivative $dP_{\mu,t}/dP_{0,t} = Y(t)$.

Solution: First consider a fixed $0 < s \le t$ and $B \in \mathcal{B}$. Then

$$P_\mu(\mathbb{S}_\mu(s) \in B) = P(N(\mu s, s) \in B) = \int_B \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{(x-\mu s)^2}{2s}\right) dx \,.$$

On the other hand, since Y is a martingale,

$$\begin{aligned}\int_{[\mathbb{S}(s)\in B]} Y(t) dP_0 &= \int_{[\mathbb{S}(s)\in B]} Y(s) dP_0 \\ &= \int_B \exp(\mu x - \frac{1}{2}\mu^2 s) \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{x^2}{2s}\right) dx \\ &= \int_B \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{(x-\mu s)^2}{2s}\right) dx \,.\end{aligned}$$

Thus

$$P_\mu(\mathbb{S}_\mu(s) \in B) = \int_{[\mathbb{S}(s)\in B]} Y(t) dP_0 = \int_{[\mathbb{S}(s)\in B]} Y(s) dP_0 \,.$$

Now $\mathbb{S}_\mu(t) - \mathbb{S}_\mu(s) \sim N(\mu(t-s), t-s)$, so (a) continues to hold for $\mathbb{S}_\mu(t) - \mathbb{S}_\mu(s)$: i.e. for $B \in \mathcal{B}$,

$$\begin{aligned}P_\mu(\mathbb{S}_\mu(t) - \mathbb{S}_\mu(s) \in B) &= \int_{[\mathbb{S}_\mu(t)-\mathbb{S}_\mu(s)\in B]} [Y(t)/Y(s)] dP_0 \\ &= \int_{[\mathbb{S}_\mu(t)-\mathbb{S}_\mu(s)\in B]} \exp(\mu(\mathbb{S}(t) - \mathbb{S}(s)) - \mu^2(t-s)) dP_0 \,.\end{aligned}$$

Now let $m \ge 1$, $t_{mi} = it/2^m$, $i = 0, 1, \ldots, 2^m$, and set

$$X_i \equiv \mathbb{S}_\mu(t_{mi}) - \mathbb{S}_\mu(t_{m,i-1}) \,,$$

and for $B_i \in \mathcal{B}$, $i = 1, \ldots, 2^m$, let

$$D_\mu \equiv \{\mathbb{S}_\mu(t_{mi}) - \mathbb{S}_\mu(t_{m,i-1}) \in B_i,\ i = 1, \ldots, 2^m\} = \cap_{i=1}^{2^m} \{\mathbb{S}_\mu(t_{mi}) - \mathbb{S}_\mu(t_{m,i-1}) \in B_i\}\,.$$

Now the X_i's are i.i.d. $N(\mu t/2^m, t/2^m)$, so that

$$\begin{aligned}
P_\mu(D_\mu) &= P_\mu(\mathbb{S}_\mu(t_{mi}) - \mathbb{S}_\mu(t_{m,i-1}) \in B_i,\ i = 1, \ldots, 2^m) \\
&= \prod_{i=1}^{2^m} P_\mu(X_i \in B_i) \\
&= \prod_{i=1}^{2^m} \int_{[X_i \in B_i]} \exp(\mu(\mathbb{S}(t_{mi}) - \mathbb{S}(t_{m,i-1})) - (1/2)\mu^2 t/2^m) dP_0 \qquad \text{by (b)} \\
&= \int_D \prod_{i=1}^{2^m} \exp(\mu(\mathbb{S}(t_{mi}) - \mathbb{S}(t_{m,i-1})) - (1/2)\mu^2 t/2^m) dP_0 \\
&\qquad \text{by independence of the } X_i's \\
&= \int_D \exp(\mu \mathbb{S}(t) - (1/2)\mu^2 t) dP_0 \\
&= \int_D Y(t) dP_0\,.
\end{aligned}$$

But the sigma - field generated by all events of the form D for $B_i \in \mathcal{B}$ and $m \geq 1$ is just $\mathcal{A}_t$. Hence it follows from the equality in the last display that for $A \in \mathcal{A}_t$,

$$P_\mu(A) = \int_A Y(t) dP_0\,.$$

Thus $Y(t) = dP_{\mu,t}/dP_{0,t}$.

5. Suppose that $\mathbb{S}_\mu$ is Brownian motion with drift $\mu > 0$ as in problem 2, and let $\tau \equiv \inf\{t > 0 : \mathbb{S}_\mu(t) = a\}$, $a > 0$. Use the result of problem 2 together with results from class concerning the distribution of τ when $\mu = 0$ to find the distribution of τ when $\mu > 0$. You should find that

$$\begin{aligned}
P_\mu(\tau > t) &= P_\mu(\mathbb{S}_\mu(s) < a,\ 0 \leq s \leq t) \\
&= \Phi\left(\frac{a - \mu t}{\sqrt{t}}\right) - e^{2\mu}\Phi\left(\frac{-a - \mu t}{\sqrt{t}}\right)
\end{aligned}$$

and

$$f_\tau(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{(a-\mu t)^2}{2t}\right) \qquad \text{for } t \geq 0\,.$$

This is the *inverse Gaussian* density. [Note that this reduces to the density of τ_a from class when $\mu = 0$.]

Solution 1: Here is a solution using the martingale property of Y which involves almost no calculation:

$$\begin{aligned}
P_\mu(\tau_\mu \leq t) &= \int_{[\tau_\mu \leq t]} dP_\mu = \int_{[\tau \leq t]} Y(t) dP_0 \\
&= \int_{[\tau \leq t]} Y(\tau) dP_0 \\
&\qquad \text{by optional sampling since } \tau \text{ is } \mathcal{A}_\tau - \text{measurable} \\
&= \int_{[\tau \leq t]} \exp(\mu a - (1/2)\mu^2 \tau) dP_0 \qquad \text{since } \mathbb{S}_\tau) = a \\
&= \int_{[s \leq t]} \exp(\mu a - (1/2)\mu^2 s) \frac{a}{\sqrt{2\pi s^3}} \exp\left(-\frac{a^2}{2s}\right) ds \\
&\qquad \text{by the computation of the density of } \tau \text{ under } P_0 \\
&\qquad \text{from problem set \#6} \\
&= \int_0^t \frac{a}{\sqrt{2\pi s^3}} \exp\left(-\frac{(a-\mu s)^2}{2s}\right) ds\,.
\end{aligned}$$

Thus under P_μ the stopping time $\tau = \tau_a$ has the *inverse Gaussian* density

$$f_\tau(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{(a-\mu t)^2}{2t}\right)\,, \quad t > 0\,.$$