Statistics 523, Problem Set 5 Solutions

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1. Suppose that $\{X(t) : t \ge 0\}$ is a process with stationary and independent increments with $X(0) = 0$ and characteristic function of $X(t)$given by

$$Ee^{iuX(t)} = \exp(-tc|u|^{\alpha}\{1 - i\text{sign}(u)C_{\alpha}\})$$

where $\alpha \in (0, 1)$, $c \ge 0$ and $C_{\alpha} = \tan(\pi\alpha/2)$. Thus the (marginal, or one-dimensional) distributions of $X(t)$ are completely asymmetric stable laws with exponent $\alpha \in (0, 1)$.
(a) Show that $X(t) \overset{d}{=} t^{1/\alpha}X(1)$ for all $t > 0$.
(b) Let $0 < r < \alpha$. Use (a) to compute $E|X(t)|^r$ in terms of $E|X(1)|^r$ where the latter is finite by Problem xx of problem set 3.
(c) Apply Theorem 12.2.2 to show that $X : (\Omega, \mathcal{A}, P) \to (R_{[0,1]}, \mathcal{B}_{[0,1]}, P_X)$ has an equivalent version $Z : (\Omega, \mathcal{A}, P) \to (R_{[0,1]}, \mathcal{B}_{[0,1]}, P_Z)$ which satisfies $Z : (\Omega, \mathcal{A}, P) \to (D, \mathcal{D}, P_Z)$ with $Z(t) = X(t)$ a.s. for each $t \in [0, 1]$.
(d) Is there an analogous theorem when X is viewed as a process with values in $(R_{[0,\infty)}, \mathcal{B}_{[0,\infty)}, P_X)$ and with versions in $(D_{[0,\infty)}, \mathcal{D}_{[0,\infty)})$?

Solution: (a) Now

$$Ee^{iuX(t)} = \exp(-tc|u|^{\alpha}\{1 - i\text{sign}(u)C_{\alpha}\})$$

so we compute, using $t^{1/\alpha} > 0$,

$$\begin{aligned} Ee^{iut^{1/\alpha}X(1)} &= \exp(-1 \cdot c|ut^{1/\alpha}|^{\alpha}\{1 - i\text{sign}(ut^{1/\alpha})C_{\alpha}\}) \\ &= \exp(-tc|u|^{\alpha}\{1 - i\text{sign}(u)C_{\alpha}\}) \\ &= Ee^{iuX(t)}, \end{aligned}$$

and hence we conclude, by the uniqueness of characteristic functions, that $X(t) \overset{d}{=} t^{1/\alpha}X(1)$.
(b) It follows from (a) that for $r < \alpha$ we have

$$E|X(t)|^r = E|t^{1/\alpha}X(1)|^r = t^{r/\alpha}E|X(1)|^r \equiv M_r t^{r/\alpha}$$

with $M_r \equiv E|X(1)|^r < \infty$.
(c) To apply Theorem 12.2.2 we first compute, for $0 \le r \le s \le t \le 1$ and $b < \alpha$,

$$\begin{aligned} E|X(r,s]X(s,t]|^b &= E\{|X(r,s]|^b\} \cdot E\{|X(r,s]|^b\} \quad \text{since } X \text{ has independent increments} \\ &= E\{|X(s-r)|^b\} \cdot E\{|X(t-s)|^b\} \text{ since } X \text{ has stationary increments} \\ &\le M_b(s-r)^{b/\alpha} M_b(t-s)^{b/\alpha} \end{aligned}$$

where we want to choose $a \equiv b/\alpha > 1/2$. But this holds if $\alpha > b > \alpha/2$. Thus Theorem 12.2.2 applies with $F(t) \equiv t$ on $[0,1]$, $K = M_b^2$, and $b \in (\alpha/2, \alpha)$, and we conclude that there is a version of the process X with sample paths in $(D, \mathcal{D})$.
(d)

2. Now let $\mathbb{S}$ be a standard Brownian motion on $[0,\infty)$, let $X(t)$ be a completely asymmetric stable process (sometimes called a *stable subordinator*) of index $\alpha \in (0,1)$ as in problem 1 above which is independent of $\mathbb{S}$. Consider the new process $Y(t) \equiv \mathbb{S}(X(t))$ for $t \ge 0$.
(a) Use a calculation similar to that of problem 4, Problem set 3, to show that Y is a symmetric stable process of index 2α.
(b) Does Y have stationary independent increments?

Solution: (a) We calculate conditionally on X to compute

$$\begin{aligned} E\exp(iuY(t)) &= EE\{\exp(iu\mathbb{S}(X(t)))|X(t)\} = E\exp(-u^2X(t)/2) \\ &= E\exp(-u^2t^{1/\alpha}X(1)/2) \;\text{ by using (a) of problem 1 above} \\ &= \psi(u^2t^{1/\alpha}/2) \text{ where } \psi \text{ is the Laplace transform of } X(1) \\ &= \exp\{-c(u^2t^{1/\alpha}/2)^\alpha\} \\ &= \exp\{-2^\alpha c \cdot t \cdot |u|^{2\alpha}\}. \end{aligned}$$

Thus we see that the marginal distributions of Y are those of a symmetric stable process of index 2α.
(b) Now for $0 \le s \le t$ we have $Y(t) - Y(s) = \mathbb{S}(X(t)) - \mathbb{S}(X(s))$ which is independent of $\{\mathbb{S}(X(r)) : 0 \le r \le s\}$ since X is monotone non-decreasing. Furthermore

$$\begin{aligned} Y(t) - Y(s) &= \mathbb{S}(X(t)) - \mathbb{S}(X(s)) \overset{d}{=} \mathbb{S}(X(t) - X(s)) \\ &\qquad \text{since } \mathbb{S} \text{ has stationary increments} \\ &\overset{d}{=} \mathbb{S}(X(t-s)) = Y(t-s) \end{aligned}$$

since X has stationary increments. Thus the process Y has stationary and independent increments.

3. Let $\tau = \tau_{ab}$ of Theorem 12.6.1; i.e. $\tau_{a,b} = \inf\{t : \mathbb{S}(t) \in (-a, b)^c\}$ for $a, b > 0$. Show that $E(\tau^2) \le 4ab(a+b)^2$. (This is slightly different from the statement of (4) in Theorem 12.6.1 on page 319, but seems to be consistent with (5) on the same page with $r = 2$. My current computation yields, in fact, $E(\tau^2) \le 2ab(a+b)^2$. *Hint:* use the martingale $\mathbb{S}^4(t) - 6t\mathbb{S}^2(t) + 3t^2$ (which follows from considering the 4th derivative with respect to θ of the exponential martingale $V_\theta(t) = \exp(\theta\mathbb{S}(t) - \theta^2 t^2/2)$ at $\theta = 0$; see Exercise 12.7.3, page 325.

   Solution: Now $V_\theta(t) \exp(\theta\mathbb{S}(t) - \theta^2 t/2)$ is a martingale for every θ. Hence $Y_k(t) \equiv \frac{\partial^k}{\partial\theta^k} V_\theta(t)\big|_{\theta=0}$ is also a martingale. We use the martingale $Y_4(t) \equiv \mathbb{S}^4(t) - 6t\mathbb{S}^2(t) + 3t^2$. Since $\tau = \tau_{ab}$ is a stopping time, it follows from the optional sampling theorem that

$$0 = EY_r(\tau) = E\mathbb{S}^4(\tau) - 6E(\tau\mathbb{S}^2(\tau)) + 3E(\tau^2).$$

   Rearranging this yields

$$\begin{aligned}
3E\tau^2 &= 6E(\tau\mathbb{S}^2(\tau)) - E\mathbb{S}^4(\tau) \\
&\le 6E(\tau(b \vee a)^2) - (b^4 \cdot \frac{a}{a+b} + a^4 \cdot \frac{b}{a+b} \\
&\le 6(a+b)^2 E\tau - \frac{ab}{a+b}(a^3 + b^3) \\
&\le 6(a+b)^2 ab - 2^{-2}\frac{(a+b)^3}{a+b} \\
&\qquad \text{by the } c_r \text{ inequality with } r = 3: \quad (a+b)^3 \le 2^{3-1}(a^3 + b^3) \\
&= (6 - 1/4)ab(a+b)^2,
\end{aligned}$$

   and hence $E\tau^2 \le 2ab(a+b)^2$.

4. PfS Course Notes, Exercise 11.1.1, page 274: 12.10.1 page 338: Suppose that $Y_1, Y_2, \ldots$ are i.i.d. Exponential(1) random variables. Set $\eta_{n,j} \equiv \sum_{i=1}^{j} Y_i / \sum_{i=1}^{n+1} Y_i$ and let $\eta_n \equiv (\eta_{n,1}, \ldots, \eta_{n,n})$. Show that $\eta_n \overset{d}{=} (\xi_{n:1}, \ldots, \xi_{n:n})$ where $0 \le \xi_{n:1} \le \cdots \le \xi_{n:n} \le 1$ are the order statistics of $\xi_1, \ldots, \xi_n$ i.i.d. Uniform$(0,1)$ random variables.

Solution: The joint density of $Y_1, \ldots, Y_{n+1}$ is given by

$$f_{\underline{Y}}(y_1, \ldots, y_{n+1}) = \exp\left(-\sum_{j=1}^{n+1} y_j\right) 1\{y_1 > 0, \ldots, y_{n+1} > 0\}.$$

Now the joint density of $W_k \equiv \sum_{j=1}^{k} Y_j$ for $k = 1, \ldots, n+1$ is given by

$$f_{\underline{W}}(w_1, \ldots, w_{n+1}) = \exp(-w_{n+1}) 1\{0 < w_1 < w_2 < \cdots < w_{n+1}\};$$

here we used $Y_k = W_k - W_{k-1}$ for $k = 1, \ldots, n+1$ so that the Jacobian of the transformation is 1. Then we transform to $\eta_{n,k} \equiv Z_k \equiv W_k / W_{n+1}$ for $k \in \{1, \ldots, n\}$ and $Z_{n,n+1} \equiv W_{n+1}$. The joint density of $\underline{Z} = (Z_1, \ldots, Z_n, Z_{n+1})$ is given by

$$f_{\underline{Z}_{n+1}}(z_1, \ldots, z_n, z_{n+1}) = z_{n+1}^n \exp(-z_{n+1}) 1\{0 < z_1 < \cdots < z_n < 1, z_{n+1} > 0\}$$

since the Jacobian of the transformation is z_{n+1}^n. Thus $(\eta_{n,1}, \ldots, \eta_{n,n}) = (Z_1, \ldots, Z_n) \equiv \underline{Z}_n$ is independent of $Z_{n+1} \equiv W_{n+1}$ with $Z_{n+1} \sim \text{Gamma}(n+1, 1)$ and the joint density of $\underline{Z}_n$ is

$$f_{\underline{Z}_n}(z_1, \ldots, z_n) = n! 1\{0 < z_1 < \cdots < z_n < 1\},$$

which is the joint density of the order statistics of a sample of n Uniform$(0,1)$ random variables. [This is related to Basu's theorem: if the Y's were exponential(θ), then $\sum_{j=1}^{n+1}$ is sufficient for θ, and the distribution of $\underline{Z}_n$ is ancillary (i.e. has a distribution which does not depend on θ), and by Basu's theorem Z_{n+1} and $\underline{Z}_n$ are independent.]