Statistics 523, Problem Set 4 Solutions

Wellner; 5/1/2013

1. PfS Course Notes, Exercise 12.3.1, page 309:
Let $Z \sim N(0,1)$ and the Brownian bridges $\mathbb{V}$, $\mathbb{U}^{(1)}$, and $\mathbb{U}^{(2)}$ be independent. Fix $a > 0$. Show that:
(11) $\mathbb{S}(t) = \mathbb{V}(t) + tZ$, $0 \le t \le 1$ is a Brownian motion.
(12) $\mathbb{S}(at)/\sqrt{a}$, $0 \le t < \infty$ is a Brownian motion.
(13) $\mathbb{S}(t+a) - \mathbb{S}(a)$, $t \ge 0$, is a Brownian motion.
(14) $\sqrt{1-a}\,\mathbb{U}^{(1)} \pm \sqrt{a}\,\mathbb{U}^{(2)}$ is a Brownian bridge if $0 \le a \le 1$.
(15) $\mathbb{Z}(t) \equiv \{\mathbb{U}^{(1)}(t) + \mathbb{U}^{(2)}(1-t)\}/\sqrt{2}$, $0 \le t \le 1/2$ is a Brownian bridge.
(16) $\mathbb{U}(t) = (1-t)\mathbb{S}(t/(1-t))$, $0 \le t \le 1$, is a Brownian bridge; use the LIL at infinity to show that this $\mathbb{U}$ converges to 0 at $t = 1$.
(17) $t\mathbb{S}(1/t)$, $0 \le t < \infty$ is a Brownian motion; apply the LIL of (10) to verify that these sample paths converge to 0 at $t = 0$.

Solution: (11) Now $\mathbb{S}$ is clearly Gaussian with $E\mathbb{S}(t) = E\mathbb{V}(t) + tE(Z) = 0 + t\cdot 0 = 0$ and, by independence of $\mathbb{V}$ and Z,

$$E\mathbb{S}(s)\mathbb{S}(t) = E\mathbb{V}(s)\mathbb{V}(t) + stE(Z^2) = s\wedge t - st + st = s \wedge t$$

for $0 \le s, t \le 1$, so $\mathbb{S}$ is a Brownian motion process on $[0,1]$.
(12) Again $\tilde{\mathbb{S}} \equiv \mathbb{S}(at)/\sqrt{a}$ is a Gaussian process with $E\mathbb{S}(t) = 0$ and $E\tilde{\mathbb{S}}(s)\tilde{\mathbb{S}}(t) = \{(as)\wedge(at)\}/a = s \wedge t$, so again $\tilde{\mathbb{S}}$ is a Brownian motion process on $[0,\infty)$.
(13) $\tilde{\mathbb{S}}(t) \equiv \mathbb{S}(t+a) - \mathbb{S}(a)$ is clear Gaussian with mean 0 and

$$\begin{aligned} E\tilde{\mathbb{S}}(s)\tilde{\mathbb{S}}(t) &= E\{(\mathbb{S}(s+a)-\mathbb{S}(a))(\mathbb{S}(t+a)-\mathbb{S}(a))\} \\ &= (s+a)\wedge(t+a) - (s+a)\wedge a - (t+a)\wedge a + a \\ &= (s+a) - a - a + a = s \quad \text{if } \ s \le t, \\ &= s\wedge t \ \text{ for arbitrary } \ s,t \ge 0. \end{aligned}$$

Hence $\tilde{\mathbb{S}}$ is again Brownian motion on $[0,\infty)$.
(14) Now $\mathbb{V}(t) \equiv \sqrt{1-a}\,\mathbb{U}^{(1)} \pm \sqrt{a}\,\mathbb{U}^{(2)}$ with $\mathbb{U}^{(1)}$ and $\mathbb{U}^{(2)}$ independent

and $a \in [0,1]$ is clearly Gaussian with mean 0 since both $\mathbb{U}^{(1)}$ and $\mathbb{U}^{(2)}$ are Gaussian with 0 mean. Furthermore

$$\begin{aligned} E\mathbb{V}(s)\mathbb{V}(t) &= (1-a)E\mathbb{U}^{(1)}(s)\mathbb{U}^{(1)}(t) + aE\mathbb{U}^{(1)}(s)\mathbb{U}^{(1)}(t) \\ &= (1-a)(s \wedge t - st) + a(s \wedge t - st) = s \wedge t - st. \end{aligned}$$

Thus $\mathbb{V}$ is a standard Brownian bridge process on $[0,1]$.
(15) Again, $\mathbb{Z}$ is a mean 0 Gaussian process on $[0,1]$ and

$$\begin{aligned} E\{\mathbb{Z}(s)\mathbb{Z}(t)\} &= 2^{-1}\{E\mathbb{U}^{(1)}(s)\mathbb{U}^{(1)}(t) + E\mathbb{U}^{(2)}(1-s)\mathbb{U}^{(2)}(1-t) \\ &= 2^{-1}\{s \wedge t - st + (1-s) \wedge (1-t) - (1-s)(1-t)\} \\ &= 2^{-1}\{s - st + (1-t) - (1-s)(1-t)\} \text{ if } \quad s \le t \\ &= 2^{-1}\{s - st + (1-t)(1-(1-s))\} = s(1-t) \quad \text{if } \; s \le t \\ &= s \wedge t - st \quad \text{for } \; 0 \le s,t \le 1. \end{aligned}$$

Thus $\mathbb{Z}$ is a standard Brownian bridge process.
(16) Now $\mathbb{U}$ is a mean zero Gaussian process since $\mathbb{S}$, and we compute

$$\begin{aligned} E\mathbb{U}(s)\mathbb{U}(t) &= E\{(1-s)\mathbb{S}(s/(1-s))(1-t)\mathbb{S}(t/(1-t))\} \\ &= (1-s)(1-t)\frac{s}{1-s} \wedge \frac{t}{1-t} \\ &= (1-s)(1-t)\frac{s}{1-s} \quad \text{if } \; s \le t \\ &= s(1-t) \quad \text{if } \; s \le t \\ &= s \wedge t - st \;\text{ for general } \; 0 \le s,t \le 1. \end{aligned}$$

Moreover, note that

$$\begin{aligned} &\lim_{t \nearrow 1}(1-t)|\mathbb{S}(t/(1-t))| \\ &\le \lim_{t \nearrow 1}(1-t)\sqrt{2(t/(1-t)\log\log(t/(1-t))}\limsup_{t \to 1}\frac{|\mathbb{S}(t/(1-t))|}{\sqrt{2(t/(1-t))\log\log(t/(1-t)))}} \\ &= \lim_{t \nearrow 1}\sqrt{2(t(1-t)\log\log(t/(1-t))}\limsup_{s \to \infty}\frac{|\mathbb{S}(s)|}{\sqrt{2s\log\log s}} \\ &= 0 \cdot 1 = 0 \;\text{ almost surely} \end{aligned}$$

by the LIL for BM at ∞. Hence $\mathbb{U}$ is a standard Brownian bridge process on $[0,1]$.

(17) Now $\tilde{\mathbb{S}}(t) \equiv t\mathbb{S}(1/t)$ is a 0−mean Gaussian process with

$$\begin{aligned} E\{\tilde{\mathbb{S}}(s)\tilde{\mathbb{S}}(t)\} &= stE\{\mathbb{S}(1/s)\mathbb{S}(1/t)\} \\ &= st((1/s) \wedge (1/t)) = st \cdot (1/t) \quad \text{if } \ s \le t \\ &= s \quad \text{if } \ s \le t \\ &= s \wedge t \ \ \text{for any } \ 0 \le s, t < \infty. \end{aligned}$$

Furthermore,

$$\begin{aligned} \lim_{t \searrow 0} |\tilde{\mathbb{S}}(t)| &\le \lim_{t \searrow 0} t\sqrt{2t \log\log(1/t)} \cdot \limsup_{t \searrow 0} \frac{|\mathbb{S}(1/t)|}{\sqrt{2t \log\log(1/t)}} \\ &= \lim_{t \searrow 0} \sqrt{2t \log\log(1/t)} \cdot \limsup_{t \searrow 0} \frac{|\mathbb{S}(1/t)|}{\sqrt{2(1/t) \log\log(1/t)}} \\ &= \lim_{t \searrow 0} \sqrt{2t \log\log(1/t)} \cdot \limsup_{r \nearrow \infty} \frac{|\mathbb{S}(r)|}{\sqrt{2(r) \log\log(r)}} \\ &= 0 \cdot 1 = 0 \quad \text{a.s.} \end{aligned}$$

by the LIL for $\mathbb{S}$ at $+\infty$. Thus $\tilde{\mathbb{S}}$ is standard Brownian motion on $[0, \infty)$.

2. PfS Course Notes, Exercise 12.3.6, page 310:
(a) Suppose that h and $\tilde{h}$ on $[0,1]$ are in $L_2[0,1]$. View *white noise* as an operator $d\mathbb{S}$ that takes the function h into a random variable $\int_{[0,1]} h(t)d\mathbb{S}(t)$ in the sense of $\to_{L_2}$. Define this integral first for step functions, and then use Exercise 4.4.6 to define it in general. Then show that $\int_{[0,1]} h(t)d\mathbb{S}(t)$ exists as such an L_2−limit for all $h \in L_2[0,1]$.
(b) In case h has a bounded derivative h'on $[0,1]$, show that

$$\int_{[0,1]} h(t)d\mathbb{S}(t) = h\mathbb{S}\Big|_{0+}^{1-} - \int_{[0,1]} \mathbb{S}(t)h'(t)dt.$$

(c) Determine the joint distribution of $\int_{[0,1]} h(t)d\mathbb{S}(t)$ and $\int_{[0,1]} \tilde{h}(t)d\mathbb{S}(t)$.

Solution: (a) First consider $h(t) = 1_{(a,b]}(t)$ for $0 \le a < b \le 1$. Then we define $\mathbb{S}(h) \equiv \int_{[0,1]} hd\mathbb{S} = \int_{(a,b]} d\mathbb{S}(t) = \mathbb{S}(b) - \mathbb{S}(a)$. Now for any function h of the form $h_m(t) = \sum_{j=1}^{m} c_j 1_{(t_{j-1},t_j]}(t)$ with c_j real and

$\sum_{j=1}^{m}(t_{j-1}, t_j] = (0,1]$, define

$$\mathbb{S}(h_m) \equiv \int_{[0,1]} h_m d\mathbb{S} = \sum_{j=1}^{m} c_j(\mathbb{S}(t_j) - \mathbb{S}(t_{j-1})).$$

Then $\mathbb{S}(h_m)$ is clearly Gaussian and we have

$$E(\mathbb{S}(h_m)^2) = \sum_{j=1}^{m} c_j^2(t_j - t_{j-1}) = \int_0^1 h_m^2(t)dt = \|h_m\|^2$$

where $\|h_m\|_2$ denotes the norm in $L_2([0,1],\lambda)$ for Lebesgue measure λ. Thus the map $h \to \mathbb{S}(h) = \int_{[0,1]} hd\mathbb{S}$ defines a linear isometry from the collection of step functions in $L_2([0,1],\lambda)$ into $L_2(\Omega, \mathcal{A}, P)$. Since the set of step functions is dense in $L_2([0,1],\lambda)$ by Exercise 4.4.6, we may extend this isometry by continuity to all $h \in L_2([0,1],\lambda)$, and by linearity the whole process $\{\mathbb{S}(h) : h \in L_2([0,1])\}$ is Gaussian. Note that for $g, h \in L_2[0,1]$ we have

$$4\langle g, h\rangle = \|g+h\|^2 - \|g-h\|^2$$

(this is known as a *polarization identity*), and hence inner products are also preserved: for all $g, h \in L_2[0,1]$

$$E(\mathbb{S}(g)\mathbb{S}(h)) = \langle g, h\rangle = \int_0^1 ghd\lambda.$$

(b) When h has a bounded derivative h', we instead define

$$\begin{aligned}\mathbb{S}(h) &\equiv h(t)\mathbb{S}(t)\Big|_{0+}^{1-} - \int_0^1 h'(u)\mathbb{S}(u)du \\ &= h(1)\mathbb{S}(1) - \int_0^1 h'(u)\mathbb{S}(u)du\end{aligned}$$

since $|h(t)| = |\int_0^t h'(s)ds| \le \|h'\|_\infty t$ for $0 \le t \le 1$. Our goal will be to show for this alternative definition that we also have $E(\mathbb{S}(h)^2) = \int_0^1 h^2(t)dt$ and hence the same isometry holds as in (a) for the functions in $C^1[0,1]$ as a subset of $L_2[0,1]$. Since $C^1[0,1]$ is also dense in $L_2[0,1]$, the definition agrees with the definition given in (a). We calculate as follows. First note that it suffices by linearity to consider h monotone non-decreasing and bounded by $\|h'\|_\infty$. For monotone h we compute (using Fubini's theorem frequently)

$$
\begin{aligned}
E\mathbb{S}^2(h) &= E\left(h(1)\mathbb{S}(1) - \int_0^1 \mathbb{S}(s)h'(s)dx\right)^2 \\
&= h^2(1) - 2h(1)E\{\mathbb{S}(1)\int_0^1 h'(s)\mathbb{S}(s)ds\} + E\left(\int_0^1 h'\mathbb{S}d\lambda\right)^2 \\
&= h^2(1) - 2h(1)\int_0^1 sh'(s)ds + \int_0^1\int_0^1 (s\wedge t)h'(s)h'(t)dsdt \\
&= h^2(1) - 2h(1)\int_0^1 sh'(s)ds + \left(\int_0^1 sh'(s)ds\right)^2 \\
&\qquad + \int_0^1\int_0^1 (s\wedge t - st)h'(s)h'(t)dsdt \\
&= \left(h(1) - \int_0^1 sh'(s)ds\right)^2 + \int_0^1\int_0^1 (s\wedge t - st)h'(s)h'(t)dsdt \\
&= \left(\int_0^1 h(s)ds\right)^2 + \int_0^1\int_0^1 (s\wedge t - st)dh(s)dh(t) \\
&= \left(\int_0^1 h(s)ds\right)^2 + Var(h(\xi)) \quad \text{where } \xi \sim U(0,1) \\
&\qquad \text{by using (6.4.13) on page 113 of PfS, course notes} \\
&= \left(\int_0^1 h(s)ds\right)^2 + \int_0^1 h^2(t)dt - \left(\int_0^1 h(t)dt\right)^2 \\
&= \int_0^1 h^2(t)dt.
\end{aligned}
$$

(c) Now for any step functions g_m, h_m in $L_2[0,1]$ with $g_m \to_2 g$ and $h_m \to_2 h$ we see that

$$\int_{[0,1]} g_m d\mathbb{S} = \sum_{j=1}^m c_j(\mathbb{S}(t_j) - \mathbb{S}(t_{j-1})) \sim N(0, \|g_m\|^2)$$

by the independent and stationary increments property of $\mathbb{S}$, and similarly

$$
\begin{aligned}
E\left(\int_{[0,1]} g_m d\mathbb{S}\right)\left(\int_{[0,1]} h_m d\mathbb{S}\right) &= \sum_{j=1}^{m} c_j d_j (t_j - t_{j-1}) = \int_0^1 g_m(t) h_m(t) dt \\
&= \langle g_m, h_m \rangle \to \langle g, h \rangle
\end{aligned}
$$

and we conclude (again) that

$$E(\mathbb{S}(g)\mathbb{S}(h)) = \langle g, h \rangle = \int_0^1 gh d\lambda.$$

Thus $(\mathbb{S}(g), \mathbb{S}(h)) \sim N_2(0, \Sigma)$ where

$$\Sigma = \begin{pmatrix} \|g\|^2 & \langle g, h \rangle \\ \langle g, h \rangle & \|h\|^2 \end{pmatrix}.$$

3. PfS Course Notes, Exercise 12.9.2, page 332:
Let $V_n(r) \equiv \sum_{k=1}^{n} |\mathbb{S}(t_{nk}) - \mathbb{S}(t_{n,k-1})|^r$ and $\|\mathcal{P}_n\| \equiv \sup_{1 \le k \le n} |t_{nk} - t_{n,k-1}|$.
(α) Let $Z \sim N(0,1)$. Let $r > 0$. Show that:
 (a) $C_r \equiv E|Z|^r = 2^{r/2}\Gamma((r+1)/2)/\sqrt{\pi}$.
 (b) $E|\mathbb{S}(t_{n,k-1}, t_{n,k}]|^r = C_r |t_{n,k} - t_{n,k-1}|^{r/2}$,
 $Var(|\mathbb{S}(t_{n,k-1}, t_{n,k}]|^r) = (C_{2r} - C_r^2)|t_{n,k} - t_{n,k-1}|^r$.
(β) Now show that $EV_n(2) = 1$ and $Var(V_n(2)) \le (C_{2r} - C_r^2)\|\mathcal{P}_n\|$ and hence

$$\sum_{n=1}^{\infty} P(|V_n(2) - 1| \ge \epsilon) \le \epsilon^{-2}(C_{2r} - C_r^2) \sum_{n=1}^{\infty} \|\mathcal{P}_n\|.$$

(γ) Finally prove that $V_n(2) \to_{a.s.} 1$ if $\sum_{n=1}^{\infty} \|\mathcal{P}_n\| < \infty$.

Solution: (α) First, if $Z \sim N(0,1)$ and $r > 0$

$$
\begin{aligned}
E|Z|^r &= 2\int_0^{\infty} z^r (2\pi)^{-1/2} \exp(-z^2/2) dz \\
&= \sqrt{\frac{2}{\pi}} \int_0^{\infty} (2v)^{(r-1)/2} e^{-v} dv \\
&= \sqrt{\frac{2}{\pi}} 2^{(r-1)/2} \Gamma((r+1)/2) = 2^{r/2}\Gamma((r+1)/2)/\sqrt{\pi} \equiv C_r.
\end{aligned}
$$

Thus $C_{2k} = 1 \cdot 3 \cdots (2k-1)$. Then note that $\mathbb{S}(t_{n,k}) - \mathbb{S}(t_{n,k-1}) \sim N(0, t_{n,k} - t_{n,k-1})$ and hence

$$E|\mathbb{S}(t_{n,k-1}, t_{n,k}]|^r = |t_{n,k} - t_{n,k-1}|^{r/2} E|Z|^r = C_r |t_{n,k} - t_{n,k-1}|^{r/2}.$$

and

$$\begin{aligned} Var(\mathbb{S}(t_{n,k-1}, t_{n,k}]^2) &= E|\mathbb{S}(t_{n,k-1}, t_{n,k}]|^4 - \{E|\mathbb{S}(t_{n,k-1}, t_{n,k}]|^2\}^2 \\ &= (C_4 - C_2^2)|t_{n,k} - t_{n,k-1}|^2 \end{aligned}$$

as claimed (and with $r = 2$).
(β) Now it follows from (α) that

$$EV_n(2) = C_2 \sum_{k=1}^{n} (t_{n,k} - t_{n,k-1}) = 1 \cdot \sum_{k=1}^{n} (t_{n,k} - t_{n,k-1}) = 1$$

since $C_2 = 1$, and using the independence of the increments of Brownian motion,

$$\begin{aligned} Var(V_n(2)) &= (C_4 - C_2^2) \sum_{k=1}^{n} (t_{n,k} - t_{n,k-1})^2 \\ &\le (3-1)\|\mathcal{P}_n\| \sum_{k=1}^{n} (t_{n,k} - t_{n,k-1}) = 2\|\mathcal{P}_n\| \end{aligned}$$

as claimed (but with $r = 2$). Thus by Markov's inequality,

$$\sum_{n=1}^{\infty} P(|V_n(2) - 1| \ge \epsilon) \le 2\epsilon^{-2} \sum_{n=1}^{\infty} \|\mathcal{P}_n\|.$$

(γ) This last display together with the Borel-Cantelli lemma shows that $P(|V_n(2) - 1| \ge \epsilon \text{ i.o.}) = 0$ for every $\epsilon > 0$, and hence $V_n(2) \rightarrow_{a.s.} 1$ if $\sum_{n=1}^{\infty} \|\mathcal{P}_n\| < \infty$.

4. PfS Course Notes, Exercise 12.9.3, page 332: Prove theorem 9.1(b) when all $t_{nk} = k/2^n$. That is, with $V_n(r) \equiv \sum_{k=1}^{2^n} |\mathbb{S}(t_{nk}) - \mathbb{S}(t_{n,k-1})|^r$, show that $V_n(1) \rightarrow_{a.s.} \infty$ if $\|\mathcal{P}_n\| \equiv \sup_{1 \le k \le 2^n} |t_{nk} - t_{n,k-1}| \to 0$.

   Solution: By the Paley-Zygmund inequality, for any $\lambda \in (0,1)$ we have

$$P(V_n(1) \ge \lambda EV_n(1)) \ge (1-\lambda)^2 \frac{\{EV_n(1)\}^2}{EV_n^2(1)}$$

where, by (α) of the previous problem,

$$EV_n(1) = C_1 \sum_{k=1}^{2^n} |\frac{k}{2^n} - \frac{k-1}{2^n}|^{1/2} = C_1 2^n \cdot 2^{-n/2} = C_1 2^{n/2}.$$

and

$$Var(V_n(1)) = (C_2 - C_1^2) \sum_{k=1}^{2^n} |\frac{k}{2^n} - \frac{k-1}{2^n}| = C_2 2^n \cdot 2^{-n} = (1 - C_1^2).$$

It follows that

$$\begin{aligned} EV_n^2(1) &= Var(V_n(1)) + \{EV_n(1)\}^2 \\ &= (1 - C_1^2) + C_1^2 2^n \end{aligned}$$

where $EV_n(1) = C_1 2^{n/2} \to \infty$ as $n \to \infty$. Thus

$$\liminf_{n\to\infty} P(V_n(1) \geq \lambda EV_n(1)) \geq (1-\lambda)^2$$

for every $\lambda \in (0,1)$. This implies that $V_n(1) \to_{a.s.} 1$.

5. Let $\mathbb{S}$ denote a standard Brownian motion process on $[0, \infty)$, and define

$$\mathbb{X}(t) = e^{-t}\mathbb{S}(e^{2t}) \quad \text{for} \quad t \in (-\infty, \infty).$$

(a) Compute $E\mathbb{X}(t)\mathbb{X}(s)$ and $E(\mathbb{X}(t)\mathbb{X}(t+h))$.
(b) Is $\mathbb{X}$ a Gaussian process?
(c) Is $\mathbb{X}$ a stationary process?

Solution: (a) Now

$$\begin{aligned} E\mathbb{X}(t)\mathbb{X}(s) &= e^{-s}e^{-t}E\mathbb{S}(e^{2s})\mathbb{S}(e^{2t}) \\ &= e^{-s}e^{-t}\{e^{2s} \wedge e^{2t}\} = e^{-s}e^{-t}e^{2s} \quad \text{if} \quad s \leq t \\ &= e^{s-t}, \quad \text{if} \quad s \leq t, \\ &= e^{-|t-s|}, \quad \text{for general} \quad s, t \in -\infty\infty. \end{aligned}$$

(b) $\mathbb{X}$ is clearly a mean 0 Gaussian process since $\mathbb{S}$ is mean-zero and Gaussian. (c) Note that for any $t_1 < t_2 < \cdots < t_k$ with $t_j \in \mathbb{R}$ we have

$$(\mathbb{X}(t_1), \ldots, \mathbb{X}(t_k)) \sim N_k(0, \Sigma)$$

where $\Sigma = (\exp(-|t_k - t_j|))_{j,k=1}^m$. But if $h > 0$ we see that

$$(\mathbb{X}(t_1 + h), \ldots, \mathbb{X}(t_k + h)) \sim N_k(0, \Sigma)$$

as well since the new covariance matrix has entries $\exp(-|(t_j + h) - (t_k + h)|) = \exp(-|t_j - t_k|)$. Thus $\mathbb{X}$ is stationary. The process $\mathbb{X}$ is known as an *Ornstein-Uhlenbeck process*; see Breiman (1968), pages 347-350. It turns out that $\mathbb{X}$ is also a Markov process, and that if a process is Gaussian, stationary, and Markov with continuous sample paths, then it is an Ornstein-Uhlenbeck process up to its mean.