Statistics 523, Problem Set 3 Solutions

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1. (a) Give an example of a random variable Y with distribution function F on $\mathbb{R}^+ = [0, \infty)$ for which $EY^r = \infty$ for all $r > 0$.
   (b) Does your example in (a) have $Eg(Y) < \infty$ for some measurable function g with $g(y) \to \infty$ as $y \to \infty$.

   Solution: (a) Suppose that F is defined by

   $$1 - F(x) = \begin{cases} (\log x)^{-\gamma}, & x \ge e, \\ 1, & x < e, \end{cases}$$

   where $\gamma > 0$. Then if $Y \sim F$,

   $$\begin{aligned} EY^r &= \int_0^\infty r x^{r-1}(1 - F(x))dx = \int_0^e dx + \int_e^\infty r x^{r-1}(\log x)^{-\gamma} dx \\ &= \infty \quad \text{for all} \quad r > 0 \end{aligned}$$

   since $\lim x^r/(\log x)^\gamma = \infty$ for all $r, \gamma > 0$.
   (b) Consider $g(x) = (\log x)^\delta$. Then $g(x) \to \infty$ if $\delta > 0$. Moreover,

   $$\begin{aligned} Eg(Y) &= \int_e^\infty (\log x)^\delta \frac{\gamma}{x(\log x)^{\gamma+1}} dx \\ &= \gamma \int_1^\infty v^{-(1+\gamma-\delta)} dv = \frac{\gamma}{\gamma - \delta} < \infty \end{aligned}$$

   if $0 < \delta < \gamma$.

2. Let $X_1, X_2, \ldots$ be i.i.d. with a density function f that is symmetric about 0 and continuous and positive at 0. Show that

   $$\frac{1}{n}\left(\frac{1}{X_1} + \cdots + \frac{1}{X_n}\right) \to_d Y$$

   where Y has a Cauchy distribution.

   Solution: Let $Y_i \sim 1/X_i$ for $i \ge 1$. Then

   $$P(1/X_1 > x) = P(0 < X < 1/x) = \int_0^{1/x} f(y)dy \sim x^{-1} f(0) \quad \text{as} \quad x \to \infty$$

and similarly

$$P(1/X_1 < -x) = P(0 < X < 1/x) \sim x^{-1} f(0) \quad \text{as} \quad x \to \infty$$

so $P(Y > x)/P(|Y| > x) \to 1/2$ while $1 - F(y) \sim y^{-1} f(0)$; here $F = F_Y$ is the distribution function of $Y = 1/X$, while f is the density of X. Thus $Y \sim F$ is in the domain of attraction of a symmetric Cauchy distribution with $\alpha = 1$ and characteristic function of the form $\exp(-d|t|)$ for some $d > 0$,and we conclude that the asserted convergence holding for some Cauchy distribution of the form dC where $C \sim \text{Cauchy}(0, 1)$ and $d > 0$. To identify d we proceed by direct calculation of the chf of Y, much as in the symmetric example considered in class:

$$\begin{aligned}
\phi_Y(t) &= Ee^{itY} = Ee^{it/X} = \int_{-\infty}^{\infty} e^{it/x} f(x)dx \\
&= \int_{-\infty}^{\infty} \cos(t/x) f(x) dx \quad \text{by symmetry of} \quad f \\
&= 2 \int_0^{\infty} \cos(ty) f(1/y) y^{-2} dy.
\end{aligned}$$

Thus, using $2\int_0^\infty f(1/y)y^{-2}dy = 1$,

$$\begin{aligned}
1 - \phi_Y(t) &= 2 \int_0^{\infty} (1 - \cos(ty)) f(1/y) y^{-2} dy = 2t \int_0^{\infty} (1 - \cos(w)) f(t/w) w^{-2} dw \\
&\sim 2t f(0) \int_0^{\infty} (1 - \cos(w)) w^{-2} dw = t f(0)\pi
\end{aligned}$$

as $t \searrow 0$ since $\int_0^\infty (1 - \cos(w))w^{-2}dw = \pi/2$. Since $\phi_Y(-t) = \phi_Y(t)$ by symmetry of Y this yields

$$1 - \phi_Y(t) \sim \pi f(0)|t| \quad \text{as} \quad t \to 0.$$

Hence with $a_n = n$ and $S_n = \sum_{i=1}^n Y_i = \sum_{i=1}^n X_i^{-1}$,

$$\begin{aligned}
\phi_{S_n/n}(t) &= \phi_Y(t/n)^n = \left(1 - \frac{n(1 - \phi_Y(t/n))}{n}\right)^n \\
&\to \exp(-\pi f(0)|t|),
\end{aligned}$$

and it follows that $S_n/n \to_d \pi f(0) C$ where $C \sim \text{Cauchy}(0, 1)$.

3. (a) Let Y be a stable random variable with $\theta = 1$ and $0 < \alpha < 1$. Show that $P(Y \geq 0) = 1$. (b) Let Y be as in (a). By the conclusion of (a) the Laplace transform of Y, $\psi(\lambda) = E\exp(-\lambda Y)$ is well-defined. Show that $Y_1 + \cdots + Y_k \stackrel{d}{=} a_k Y + b_k$ holds with $b_k = 0$ (and $Y_1, \ldots, Y_k$ i.i.d. as Y).
(c) Show that $\psi(\lambda)^n = \psi(n^{1/\alpha}\lambda)$ and hence that $\psi(\lambda) = \exp(-c\lambda^\alpha)$ for some $c > 0$.

Solution: (a) Suppose that $P(Y \geq 0) = 1$ and that Y is stable with $\alpha \in (0,1)$ and $\theta = 1$. Then from our development of the characteristic function of an infinitely divisible random variable, the Lévy- Khintchine representation of the characteristic function of Y is of the form

$$\begin{aligned}\phi_Y(t) &= \exp\left(itc + m_1 \int_0^\infty \left(e^{itx} - 1 - it\frac{x}{1+x^2}\right)\frac{1}{x^{\alpha+1}}dx\right)\\ &= \exp(it\mu - d|t|^\alpha(1 - i\text{sign}(t)C_\alpha))\end{aligned}$$

where $d = \alpha^{-1}\Gamma(1-\alpha)m_1\cos(\pi\alpha/2)$ and $C_\alpha = \tan(\pi\alpha/2)$. Since Y is stable we have

$$\phi_Y(t)^k = \phi_Y(a_k t)e^{itb_k}$$

for all $k \geq 1$, and since $a_k = k^{1/\alpha}$ this yields

$$\exp(itk\mu - dk|t|^\alpha(1 - \text{sign}(t)C_\alpha)) = \exp(ik^{1/\alpha}t\mu - dk|t|^{1/\alpha}(1 - \text{sign}(k^{1/\alpha}t)C_\alpha))e^{itb_k},$$

for all t and all k, so we see that $k\mu = k^{1/\alpha}\mu + b_k$ or $-b_k = (k^{1/\alpha} - k)\mu$ and hence $-b_k/a_k = (1 - k^{1-1/\alpha})\mu \to \mu$ as $k \to \infty$ since $1 - 1/\alpha < 0$ for $0 < \alpha < 1$. Thus it follows from stability of Y and $Y \in \mathcal{D}_N(G_\alpha)$ that on the one hand

$$\phi_{(S_k - b_k)/a_k}(t) = \phi_Y(t) = \exp(it\mu - d|t|^\alpha(1 - \text{sign}(t)C_\alpha)),$$

while on the other hand

$$\phi_{(S_k - b_k)/a_k}(t) = Ee^{itS_k/a_k}\exp(-itb_k/a_k)$$

where $-b_k/a_k \to \mu$. It follows that

$$E\exp(itS_k/a_k) \to \exp(-d|t|^\alpha|t|^\alpha(1 - \text{sign}(t)C_\alpha)).$$

Since we started with $P(Y \geq 0) = 1$ it follows that $P(S_k \geq 0) = 1$ and hence the limiting distribution of $S_k/k^{1/\alpha}$, namely the distribution with chf $\exp(-d|t|^\alpha|t|^\alpha(1 - \text{sign}(t)\tan(\pi\alpha/2)))$, corresponds to the chf of a random variable which is non-negative with probability 1. Since this characteristic function is that of an $\alpha-$stable random variable $\tilde{Y}$ with $\mu = 0$ and $\theta = 1$, by uniqueness of characteristic functions we conclude that $P(\tilde{Y} \geq 0) = 1$.

(b) Since $Y \geq 0$ a.s. we have $\exp(-\lambda Y) \leq 1$ a.s., and hence $\psi(\lambda) = E\exp(-\lambda Y)$ is well-defined. From the considerations in the proof of (a) we have $b_k = -(k^{1/\alpha} - k)\mu = 0$ for all k.
(c) Since $a_k Y \stackrel{d}{=} Y_1 + \cdots + Y_k$ for every $k \geq 1$ with $a_k = k^{1/\alpha}$, it follows that

$$\psi(\lambda)^k = \psi(k^{1/\alpha}\lambda) \qquad \text{for all} \quad \lambda \geq 0, \quad k \geq 1.$$

Now the proof proceeds much as our proof for the identification of the measures $M^{\pm}$ in the stable case. Set $\lambda = (n/k)^{1/\alpha}$: then it follows that

$$k \log \psi((n/k)^{1/\alpha}) = \log \psi(k^{1/\alpha}(n/k)^{1/\alpha}) = \log \psi(n^{1/\alpha}) \quad \text{for all} \quad n, k \geq 1.$$

In particular, with $k = n$ this yields $n \log \psi(1) = \log \psi(n^{1/\alpha})$, and then substitution of this in the last display yields

$$k \log(\psi((n/k)^{1/\alpha})) = n \log \psi(1), \quad \text{or} \quad \log(\psi((n/k)^{1/\alpha})) = (n/k) \log \psi(1).$$

for all $n, k \geq 1$. Thus for all λ in the dense set $\{(n/k)^{1/\alpha}\}$ we have shown that

$$\log \psi(\lambda) = \lambda^{\alpha} \log \psi(1).$$

Since ψ is monotone decreasing, this implies that

$$\psi(\lambda) = \exp(-c\lambda^{\alpha}) \tag{1}$$

where $c \equiv -\log \psi(1) > 0$. Note that this agrees with the characteristic function as identified in (a): taking $-\lambda = it$ we have $\lambda = -ti = te^{-i\pi/2}$ for $t > 0$, and hence $\lambda^{\alpha} = t^{\alpha} e^{-i\pi\alpha/2}$. It follows that

$$\begin{aligned} -c\lambda^{\alpha} &= -ct^{\alpha} e^{-i\pi\alpha/2} = -t^{\alpha}\{\cos(\pi\alpha/2) - i \sin(\pi\alpha/2)\} \\ &= -c\cos(\pi\alpha/2) t^{\alpha}\{1 - i\tan(\pi\alpha/2)\} \end{aligned}$$

where $c \geq 0$. This is exactly the form of the $\alpha-$stable characteristic function with $\mu = 0$ and $\theta = 1$ as in (a).

Remark: See *Stable Non-Gaussian Random Processes* by Samorodnitsky and Taqqu, pages 13 - 20 for more about completely asymmetric stable distributions and their properties.

4. Show that if X is symmetric stable with index α and $Y \geq 0$ is an independent stable random variable with index $\beta < 1$, then $XY^{1/\alpha}$ is symmetric stable with index $\alpha\beta$.

Solution: If X has a symmetric stable distribution with index $\alpha \in (0,2]$ and $Y \geq 0$ is stable with index $\beta < 1$, then $W \equiv XY^{1/\alpha}$ has characteristic function

$$\begin{aligned} \phi_W(t) &= Ee^{itW} = Ee^{itXY^{1/\alpha}} = E\{E(e^{itXY^{1/\alpha}}|Y)\} \\ &= E\exp(-d|t|^\alpha Y) = \exp(-c(d|t|^\alpha)^\beta) \\ &= \exp(-cd^\beta |t|^{\alpha\beta}) \end{aligned}$$

by using Problem 3 at the next to last step. Here c and d are positive constants and $0 < \alpha\beta < \alpha < 2$.

5. Find a random variable Y with distribution function F having $EY^2 = \infty$ but with $F \in \mathcal{D}(Normal)$.

Solution: One solution to this is given by the density treated in the handout in class on Friday 19 April: $f(x) = cx^{-3}(\log x)^2 1_{[e,\infty)}(x)$. A family of symmetric examples of this type is given by

$$f(x) = c|x|^{-3}(\log|x|)^r 1_{[e,\infty)}(|x|)$$

with $r \geq -1$. Now for $r > -1$ we have

$$\begin{aligned} U(x) &= 2\int_e^x y^2 f_r(y)dy = 2c\int_e^x y^{-1}(\log y)^r d \\ &= 2c\int_1^{\log x} v^r dv = \frac{2c}{r+1}\{(\log x)^{r+1} - 1\} \\ &\sim \frac{2c}{r+1}(\log x)^{r+1} \quad \text{as} \quad x \to \infty. \end{aligned}$$

Thus with $A_n = dn^{1/2}(\log n)^{(r+1)/2}$ we have

$$\begin{aligned} \frac{nU_n(A_n)}{A_n^2} &\sim \frac{2cn(\log A_n)^{r+1}}{A_n^2} = \frac{2cn(\log(dn^{1/2}(\log n)^{r/2}))^{r+1}}{d^2 n(\log n)^{r+1}} \\ &\to \frac{2c}{2^{r+1}d^2} = 1 \end{aligned}$$

if $d = \sqrt{c/2^r}$. Thus we conclude that for $r > -1$

$$\frac{S_n - n\mu}{\sqrt{(c/2^r)n(\log n)^{r+1}}} \to_d Z \sim N(0,1).$$

When $r = -1$ we find that

$$U(x) = 2c\int_1^{\log x} v^{-1}dv = 2c\log\log x,$$

and then with $A_n = dn^{1/2}(\log\log n)^{1/2}$ we have

$$\begin{aligned} \frac{nU(A_n)}{A_n^2} &= \frac{2cn \log\log(dn^{1/2}(\log\log n)^{1/2}}{d^2 n \log\log n} \\ &\to 1 \end{aligned}$$

if $d = \sqrt{2c}$. Therefore in this case it follows that

$$\frac{S_n}{\sqrt{2cn\log\log n}} \to_d Z \sim N(0,1).$$

Note that if $r < -1$, then $r + 1 < 0$ and

$$U(x) = \frac{2c}{-(r+1)}\left(1 - (\log x)^{r+1}\right) \to \frac{2c}{-(r+1)} \equiv \sigma_r^2 < \infty,$$

so $E_r X^2 = Var_r(X) = \sigma_r^2 < \infty$, and $(S_n - n\mu)/\sqrt{n} \to_d N(0, \sigma_r^2)$.