Statistics 523, Problem Set 2 Solution

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1. PfS Course Notes, Exercise 11.1.1, page 274: (Chf expansions for the uan array $X_{n1}, \ldots, X_{nn}$) Consider a uan array of rv's X_{nk}.
(a) Let F_{nk} and ϕ_{nk} denote the df and the chf of X_{nk}. Show that

$$\max_{k \le n} |\phi_{nk}(t) - 1| \to 0 \ \text{ uniformly on every finite interval.}$$

(b) Set $\epsilon_n(t) \equiv \sum_{k=1}^n |\phi_{nk}(t) - 1|^2$. Show that if $X_{nk} \sim (0, \sigma_{nk}^2)$ and $\sigma_n^2 \equiv \sum_{k=1}^n \sigma_{nk}^2 \le M < \infty$ with $\max_{k \le n} \sigma_{nk}^2 \to 0$, then $\epsilon_n(t) \to 0$ uniformly in t on each finite interval.

Solution: (a) This was already done in Problem Set #1, problem 3, (a) implies (b).
(b) Note that since $\max_{k \le n} \sigma_{nk}^2 \to 0$, we know that the triangular array is uan by Markov's inequality:

$$\max_{k \le n} P(|X_{nk}| > \epsilon) \le \max_{k \le n} \sigma_{nk}^2 / \epsilon^2 \to 0$$

as $n \to \infty$. Furthermore, by inequality 9.6.1 we have

$$\max_{k \le n} |\phi_{nk}(t) - 1| \le K_{2,1} |t|^2 \max_{k \le n} E X_{nk}^2 = K_{2,1} |t|^2 \max_{k \le n} \sigma_{nk}^2 .$$

Thus it follows that

$$\begin{aligned}
\epsilon_n(t) &\equiv \sum_{k=1}^n |\phi_{nk}(t) - 1|^2 \\
&\le \max_{k \le n} |\phi_{nk}(t) - 1| \sum_{k=1}^n |\phi_{nk} - 1| \\
&\le \max_{k \le n} |\phi_{nk}(t) - 1| K_{2,1} |t|^t \sum_{k=1}^n \sigma_{nk}^2 \\
&\le \max_{k \le n} |\phi_{nk}(t) - 1| K_{2,1} T^2 M \\
&\to 0
\end{aligned}$$

for $|t| \le T$ and since $\sum_{k \le n} \sigma_{nk}^2 \le M$.

2. PfS Course Notes, Exercise 11.1.2, page 274:
(a) If ϕ is id, then $\phi(t) \neq 0$ for any t.
(b) Let ϕ and ϕ_n denote the chf of Y and of the Y_{nk}'s respectively where $Y \stackrel{d}{=} Y_{n1} + \cdots + Y_{nn}$ and the Y_{nk}'s are i.i.d. for each n. Show that these Y_{nk}'s form a uan array.
(c) If $Y_m \to_d Y$ for id random variables Y_m, then Y is id.

Solution: (a) Since ϕ is the characteristic function of Y, $\psi = |\phi|^2$ is also a characteristic function (of $Y - Y'$ with Y, Y' independent, $Y' \stackrel{d}{=} Y$. Since Y is infinitely divisible, so is $Y - Y'$ and hence $\psi(t) = |\phi(t)|^2$ satisfies $\psi(t) = |\phi(t)|^2 = |\phi_n(t)|^{2n}$ for some characteristic functions ϕ_n; the ϕ_n are the characteristic functions of $Y_{n,j}$ i.i.d. with $Y_{n,1} + \cdots + Y_{n,n} \stackrel{d}{=} Y$. But then $|\phi_n(t)| = |\phi(t)|^{1/n} \to 1\{|\phi(t)| \neq 0\} \equiv \tilde{\phi}(t)$. Since ϕ is continuous, we know that it is bounded away from zero for some neighborhood of 0, and hence the limit $\tilde{\phi}$ is 1 for all t in a neighborhood of 0. Since ϕ is continuous at 0, the Cramér-Lévy continuity theorem implies that it is a characteristic function. Since all characteristic functions are continuous we see that $\{t : |\phi(t)| = 0\} = \emptyset$. Thus $\phi(t) \neq 0$ for any t.
(b) That the $Y_{n,j}$'s in $Y_{n,1} + \cdots + Y_{n,n} \stackrel{d}{=} Y$ are uan essentially follows from the proof of (a) above, since the limiting characteristic function $\tilde{\phi}$ is that of the law degenerate at 0, and hence the (i.i.d. array) of $Y_{n,j}$'s is uan.
(c) Suppose that Y_m is infinitely divisible for each m and that $Y_m \to_d Y$. By infinite-divisibility of Y_m, for each $n \geq 1$ there are i.i.d random variables $\{Y_{m,n,j} : 1 \leq j \leq n\}$ such that $\sum_{j=1}^n Y_{m,n,j} \stackrel{d}{=} Y_m$. Fix n and consider the collection $\{Y_{m,n,1} : m \geq 1\}$. Then the first step is to show that this collection is tight. This proceeds along the lines of the proof of our first theorem concerning infinitely divisible laws: Note that

$$\begin{aligned} P(Y_{m,n,1} > K)^n &= P(Y_{m,n,1} > K, \ldots, Y_{m,n,n} > K) \\ &\leq P(\sum_{j=1}^n Y_{m,n,j} > nK) = P(Y_m > nK), \end{aligned}$$

so, since $\{Y_m\}$ is tight (since $Y_m \to_d Y$),

$$\limsup_{m\to\infty} P(Y_{m,n,1} > K) \leq \limsup_{m\to\infty} P(Y_m > nK) \to 0$$

as $M \to \infty$, and similarly for the left tail. Thus there is a subsequence $\{Y_{m',n,1}\}$ satisfying $Y_{m',n,1} \to_d$ some $Y_{n,1}$. Since the $Y_{m',n,k}$'s are i.i.d. for each k, this yields $Y_{n,1}, \ldots, Y_{n,n}$ i.i.d. such that $Y_{m',n,k} \to_d Y_{n,k}$ as $m' \to \infty$ for each $1 \leq k \leq n$. By the Mann-Wald theorem we find that

$$Y \stackrel{d}{=} Y_{n,1} + \cdots + Y_{n,n}.$$

Since this holds for each fixed n, Y is infinitely divisible.

3. PfS Course Notes, Exercise 11.2.1, page 283: Suppose that $a_n \nearrow$ with $a_1 = 1$ and suppose that $a_{mk} = a_m a_k$ for all $k, m \geq 1$. Then show that $a_n = n^{1/\alpha}$ for some $\alpha \geq 0$.

Solution: (From Breiman, *Probability*, page 202.) Since $a_{mk} = a_m a_k$ for all $m, k \geq 1$ it follows that $a_{k^j} = (a_k)^j$ for all $k, j \geq 1$. For $n > k$, choose j so that $a_{k^j} \leq a_n \leq a_{k^{j+1}}$. Thus we have

$$\log a_{k^j} \leq \log a_n \leq \log a_{k^{j+1}},$$

or

$$j \log a_k \leq \log a_n \leq (j+1) \log a_k.$$

Dividing by $j \log k$ yields

$$\frac{\log a_k}{\log k} \leq \frac{\log a_n}{\log n} \frac{\log n}{j \log k} \leq \frac{j+1}{j} \frac{\log a_k}{\log k}.$$

Letting $n \to \infty$ and hence also $j \to \infty$ and $\log n/(j \log k) \to 1$, yields

$$\frac{\log a_k}{\log k} = \lim_{n\to\infty} \frac{\log a_n}{\log n} \equiv \lambda > 0,$$

or equivalently $a_k = k^\lambda$.

4. PfS Course Notes, Exercise 11.2.2, page 283: Suppose that $Y \sim G$ is stable with characteristic exponent α. Then $E|Y|^r < \infty$ for all $0 < r < \alpha$. (Hint: use the inequalities of section 8.3 to show that $nP(|Y| > a_n x)$ is bounded in n where $a_n \equiv n^{1/\alpha}$, and then bound the appropriate integral.

Solution: First note that if $Z \equiv Y - Y'$ where $Y' \stackrel{d}{=} Y$ and Y, Y' are independent, then $Z$is symmetric and inequality 8.3.2 with $a = 0$ and $m \equiv \text{median}(Y)$ yields

$$\frac{1}{2} E|Y - m|^r \leq E|Z|^r \leq 2^{1+r} E|Y|^r.$$

Thus $E|Y|^r < \infty$ implies $E|Z|^r < \infty$, and conversely, via the $C_r$inequality

$$E|Z|^r \geq (1/2) E|Y - m|^r \geq \frac{1}{2}\left\{\frac{1}{C_r} E|Y|^r - m^r\right\},$$

so that $E|Z|^r < \infty$ implies $E|Y|^r < \infty$.

By Corollary 11.3.1 we have $\phi_Z(t) = \exp(-c|t|^\alpha)$. Furthermore, by inequality 9.5.1 and the elementary inequality $1 - e^{-a} \le a$ (or $1 - a \le e^{-a}$),

$$\begin{aligned} P(|Z| \ge x) &\le 7x \int_0^{1/x} (1 - \text{Re}\phi(t))dt \\ &= 7x \int_0^{1/x} (1 - \exp(-ct^\alpha)dt \le 7x \int_0^{1/x} ct^\alpha dt \\ &= 7cx \frac{1}{\alpha+1} x^{-(\alpha+1)} = \frac{7c}{\alpha+1} x^{-\alpha}. \end{aligned}$$

It follows that for $r < \alpha$

$$\begin{aligned} E|Z|^r &= \int_0^\infty r x^{r-1} P(|Z| > x) dx \le 1 + \frac{7c}{\alpha+1} \int_1^\infty x^{r-1} x^{-\alpha} dx \\ &\le 1 + \frac{7c}{\alpha+1} \frac{1}{\alpha - r} < \infty. \end{aligned}$$

Remark: It turns out that $E|Y|^\alpha = \infty$ if Y is stable with characteristic exponent $\alpha < 2$

5. Let $\gamma \ge 0$, $\delta \in \mathbb{R}$, $\delta_j \in \mathbb{R}$, and suppose that $0 < \gamma + \sum_{j=1}^\infty \delta_j^2 < \infty$. Define, for $t \in \mathbb{R}$,

$$\Psi(t) = e^{-\gamma t^2 + \delta t} \prod_{j=1}^\infty (1 + \delta_j t) e^{-\delta_j t},$$

and consider $\phi(v) \equiv 1/\Psi(iv)$.
(a) Is ϕ the characteristic function of some random variable Y? If so, identify Y in terms of some simpler independent random variables.
(b) Is Y infinitely divisible?

Solution: (a) From the definitions of Ψ and ϕ we find that

$$\phi(v) = 1/\Psi(iv) = \exp(-\gamma v^2) \exp(-iv\delta) \prod_{j=1}^\infty \frac{e^{iv\delta_j}}{(1 + \delta_j iv)}.$$

The first factor in the last display is the characteristic function of a $N(0, 2\gamma)$ random variable; i.e. $\sqrt{2\gamma} Z$ where $Z \sim N(0,1)$. The second term is the characteristic function of the constant random variable $-\delta$. Each of the factors in the infinite product is the characteristic function of $-(Y_j - \delta_j)$ where Y_j is exponential with density$\delta_j^{-1} \exp(-y/\delta_j) 1_{[y \ge 0)}$. Thus ϕ is the chf of

$$Y = \sqrt{2\gamma} Z - \delta - \sum_{j=1}^\infty (Y_j - \delta_j)$$

where the Y_j's are independent and independent of Z. Note that $E(Y_j - \delta_j) = 0$ and

$$\sum_{j=1}^{\infty} \mathrm{Var}(Y_j) = \sum_{j=1}^{\infty} \delta_j^2 < \infty$$

so that Y is almost surely a finite (proper) random variable.
(b) Y is infinitely divisible since Z is infinitely divisible, $-\delta$ is infinitely divisible and each $Y_j - \delta_j$ is infinitely divisible: for each $j \geq 1$ and $m \geq 1$ we have $Y_j - \delta_j \stackrel{d}{=} \sum_{j=1}^{m}(Y_{j,k} - \delta_j/m)$ where the $Y_{j,k}$ are i.i.d. Gamma$(1/m, \delta_j)$.